Functions of Bounded Variation: Jordan's Theorem- HW Problems

1. Suppose that $f \in BV[a,b]$ and $[c,d] \subseteq [a,b]$. Show that $f \in BV[c,d]$ and the total variation of f on $[c,d]$ is less than or equal to the total variation of f on $[a,b]$.

2. Let $f, g \in BV[a,b]$ and $c \in \mathbb{R}$. Prove

a. $TV(f) = 0$ if and only if $f(x)$ is a constant function

b. $TV(cf) = |c|TV(f)$

c. $TV(|f|) \leq TV(f)$.

3. Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ on $[a,b]$ (ie f_n converges to f pointwise).

a. Show that if each $f_n(x)$ is increasing on $[a,b]$ then so is $f(x)$ (increasing means if $y > x$ then $f(y) \geq f(x)$ for all $x, y \in [a,b]$).

b. Now suppose that instead of $f_n(x)$ being increasing, we assume $f_n \in BV[a,b]$. Show with an example that the pointwise limit of a sequence of functions of bounded variation on $[a,b]$ need not be of bounded variation on $[a,b]$.

4. Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ on $[a,b]$. Show that for any partition P of $[a,b]$, $\lim_{n\to\infty} V(f_n, P) = V(f, P)$ and if $TV(f_n) \leq M \in \mathbb{R}$, for all n, then $TV(f) \leq M$.

5. Prove that $f(x) = x^4 - 5x^3 + 4x^2 - 3x - 1$ is of bounded variation on $[1,2]$ by writing $f(x)$ as the difference of two increasing functions on $[1,2]$.