Trigonometric Polynomials: Weierstrass' 2nd Approximation Theorem- HW Problems

1. Let $T(x)$ be a trigonometric polynomial.

a. Prove that if $T(x)$ is odd (ie $T(-x) = -T(x)$) then it can be written with only sine terms.

b. Prove that if $T(x)$ is even (ie $T(-x) = T(x)$) then it can be written with only cosine terms.

2. Show that there exists a polynomial of degree $2k,\ p(x),$ such that $\sin^{2k} x = p(\cos(x))$

3. Let $T(x) = a_0 + \sum_{k=1}^{n}(a_k \cos(kx) + b_k \sin(kx))$. Show that $\int_{-\pi}^{\pi}\left(T(x)\right)^2 dx = \pi[2(a_0)^2 + \sum_{k=1}^{n}(a_k^2 + b_k^2)]$.

4. Let $f \in C^{2\pi}$ such that $\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0$ and $\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = 0$ for $n = 0,1,2,\ldots$. Show that $f(x) = 0$ (This implies that the Fourier series of a function is unique).

Hint: Consider the example done in the section on Weierstrass' Approximation Theorem that showed if $f \in C[a,b]$ and $\int_a^b x^n f(x)dx = 0;$ for all $n = 0, 1, 2, \ldots$ then $f(x) = 0$.