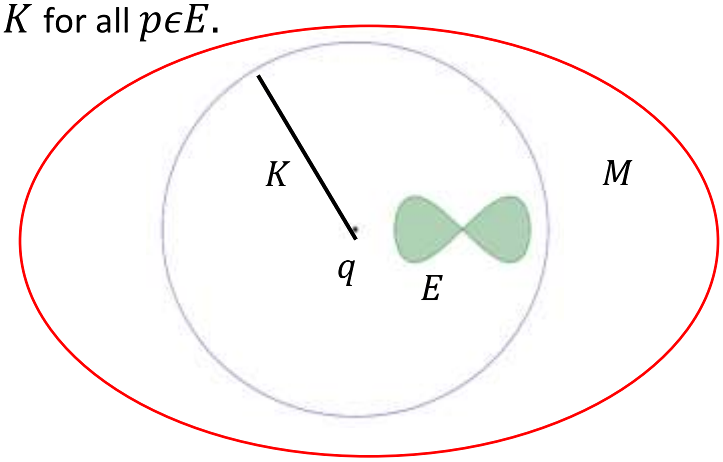
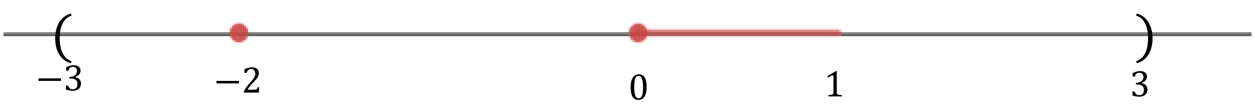
Totally Bounded Sets

Def. A set E in a metric space M, d is **bounded** if there is a real number K and a point $q \epsilon M$ such that $d(p, q) < K$ for all $p \epsilon E$.

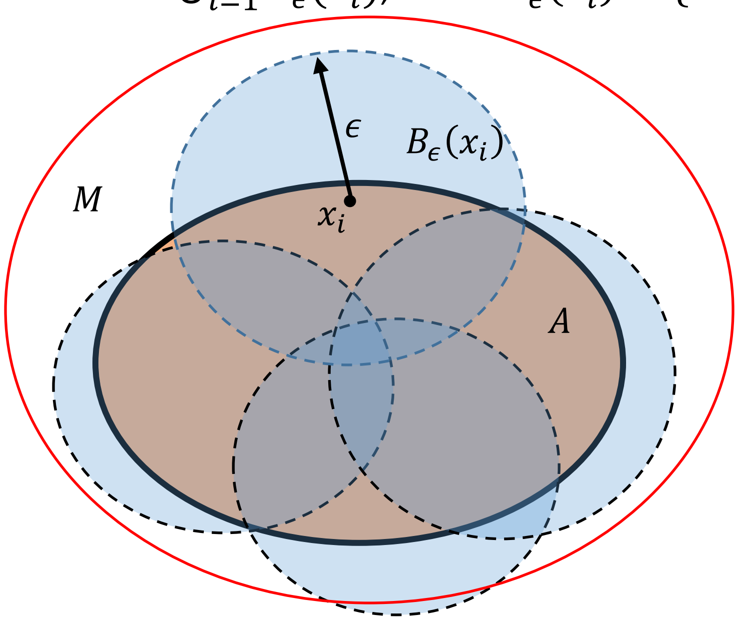


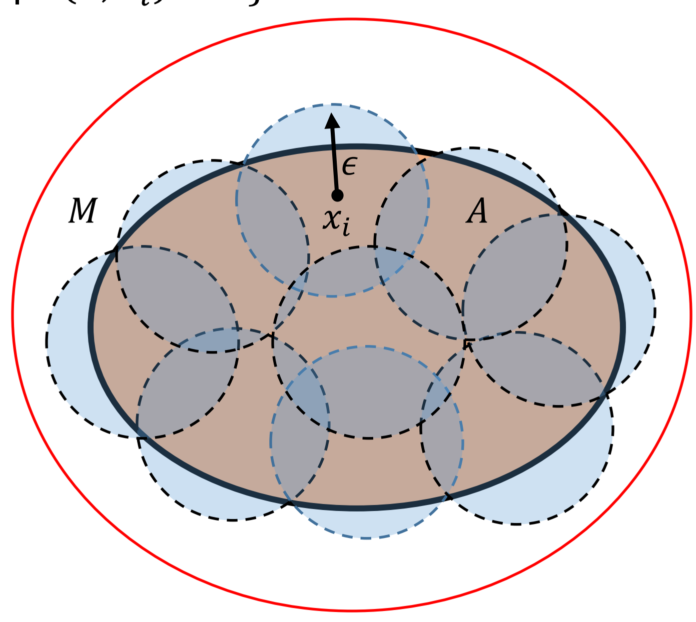
Ex. Let $M = \mathbb{R}$, and d the standard metric. Let $E = [0,1) \cup \{-2\}$.

E is a bounded set. We can take $0 \epsilon M$ and $d(0, p) < 3$, for all $p \epsilon E$.



Def. A set A in a metric space M, d is said to be **totally bounded** if, given any $\epsilon > 0$, there exist finitely many points $x_1, x_2, \ldots, x_n \in M$ such that $A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$, where $B_\epsilon(x_i) = \{x \in M \mid d(x, x_i) < \epsilon\}$.





Ex. $(-2,3] \cup \{7\} \cup [8,9]$ is a totally bounded set in $\mathbb{R}$ (In fact, in $\mathbb{R}$ being totally bounded is equivalent to being bounded. However, this is not true for a general metric space).

Ex. $(-1,6] \cup (9,\infty)$ is not a totally bounded set in $\mathbb{R}$.

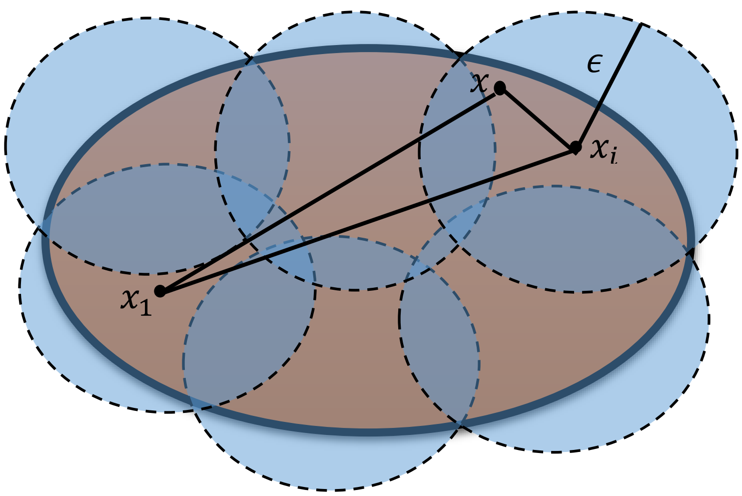
Notice that if A is totally bounded, then A is bounded (but not the other way around). Let's see why.

Given any $\epsilon > 0$, there exist finitely many points $x_1, x_2, \dots, x_n$ such that

$A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$.

If x is any point in A then $x \in B_\epsilon(x_i)$, for some i.

Now we show that $d(x, x_1) \leq \max_{1\leq k\leq n} d(x_1, x_k) + \epsilon$.



By the triangle inequality:

$$d(x, x_1) \leq d(x, x_k) + d(x_k, x_1) \leq d(x, x_k) + \max_{1\leq k\leq n} d(x_1, x_k).$$

But x has to be in $B_\epsilon(x_i)$, for some i, so $d(x, x_i) < \epsilon$. Thus we have:

$$d(x, x_1) \leq d(x, x_i) + d(x_i, x_1) < \epsilon + \max_{1 \leq k \leq n} d(x_1, x_k).$$

Thus the distance between any point in A and the point x_1 is bounded, hence A is bounded.

Now let's see an example where A is bounded but not totally bounded.

Ex. Let $M = \{sequences\ of\ real\ numbers\ \{x_i\}\ such\ that\ \sum_{i=1}^{\infty} |x_i| < \infty\}$.

We can turn this set into a metric space with the following metric:

$$d(\{x_i\}, \{y_i\}) = \sum_{i=1}^{\infty} |x_i - y_i|.$$

This metric space is usually called l_1.

Let $A = \{\{x_i\} | x_i = 1\ for\ only\ one\ i\ and\ 0\ otherwise\}$. So the elements of A are:

$e_1 = \{1,0,0,0,0, \dots\}$

$e_2 = \{0,1,0,0,0, \dots\}$

$\vdots$

$e_n = \{0,0,0, \dots, 1,0,0, \dots\}$, where the n^{th} element is 1, etc.

$\vdots$

If $z = \{0,0,0, \dots\} \in M$, then $d(z, e_i) = 1$, for all $e_i \in A$.

Thus A is bounded.

Now let's show that A is not totally bounded.

Notice that $d(e_i, e_j) = 2$ for $i \neq j$.

Thus every element of A is a distance 2 from every other element of A.

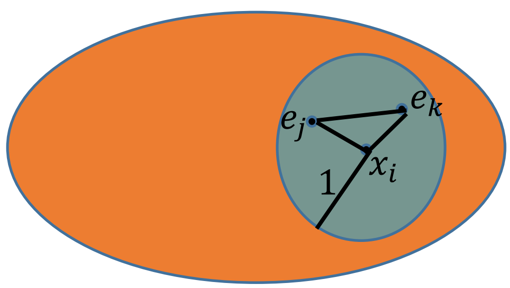
To show that A is not totally bounded we have to find an $\epsilon > 0$ and show that we can't find a finite number of points $x_1, x_2, \dots, x_n \in M$ such that $A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$.

We can do this by choosing ϵ (less than or) equal to $\frac{1}{2}$ of the common distance between all of the points of A.

So in this case $\epsilon = 1$.

Let's assume that we can find a finite number of points $x_1, x_2, \dots, x_n \in M$ such that $A \subseteq \bigcup_{i=1}^{n} B_1(x_i)$, and get a contradiction.

Since A has an infinite number of elements, then for some i, $B_1(x_i)$, must contain at least 2 elements of A (in fact, in this case there must be an i such that $B_1(x_i)$ contains an infinite number of points in A).



But by the triangle inequality, if $e_j, e_k \in B_1(x_i)$, $j \neq k$, then:

$$2 = d(e_j, e_k) \leq d(e_j, x_i) + d(x_i, e_k) < 1 + 1 = 2$$

which is a contradiction (2 can't be less than 2).

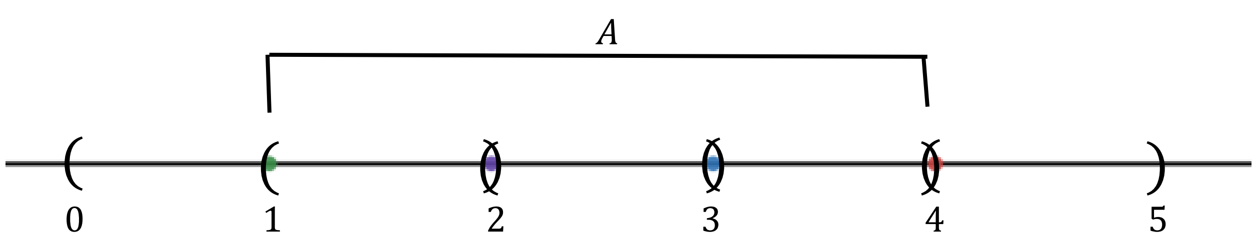
Thus there exists an $\epsilon > 0$, such that we can't find a finite number of points $x_1, x_2, \dots, x_n \in M$ where $A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$.

Thus A is not totally bounded.

In the previous example we were able to show that if a set A has an infinite number of points that are a fixed distance from every other point in the set, then A cannot be totally bounded. Notice that this can't happen in $\mathbb{R}^n$ with the standard metric. For example, if we take 3 as the fixed distance, the largest subset of $\mathbb{R}$ where every point is a distance 3 from every other point, is a subset with 2 points in it. In $\mathbb{R}^2$ the largest subset has 3 point in it. In $\mathbb{R}^n$ the largest subset where every point is a distance 3 from every other point is a set with $n+1$ elements.

In fact, every bounded set in $\mathbb{R}^n$ (with the standard metric) is totally bounded (this is left as an exercise). To see one way to approach a proof of this statement let's take $A = [1, 4] \subseteq \mathbb{R}$ and let $\epsilon = 1.$

Clearly, we can take balls of radius 1 centered at $x_1 = 1,\ x_2 = 2,\ x_3 = 3$, and $x_4 = 4$ and they will cover A.



If $\epsilon = \frac{2}{3}$ we could take balls of radius $\frac{2}{3}$ centered at $x_1 = 1,\ x_2 = \frac{5}{3}\ ,\ x_3 = \frac{7}{3}\ ,$ $x_4 = 3$, and $x_5 = \frac{11}{3}$ and they will cover A.

Generalizing this process for any given ϵ shows that A is totally bounded.

However, if we change the metric on $\mathbb{R}^n$ (or $\mathbb{R}$) a bounded set is not necessarily totally bounded.

Ex. Let $M = \mathbb{R}$ and the metric given by $d(x, y) = 8$ if $x \neq y$ and $d(x, x) = 0$. Now let $A = \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$.

a. Show that A is bounded.

b. Show that A is not totally bounded.

a. $\frac{1}{2} \in M$ and $d\left(\frac{1}{2}, x\right) = 8$ for all $x \in A$. Thus A is bounded.

b. For any two points $x, y \in A$, $d(x, y) = 8$, if $x \neq y$.

Thus take $\epsilon = \frac{8}{2} = 4$ (or any positive number smaller than 4) and suppose there exist $x_1, \dots, x_n \in M$ with $A \subseteq \bigcup_{i=1}^{n} B_4(x_i)$.

Since A has an infinite number of elements, for some i, $B_4(x_i)$ contains at least two different elements of A. Let's call there elements $p_1, p_2 \in B_4(x_i)$.

Now by the triangle inequality (and using the fact that $p_1, p_2 \in B_4(x_i)$):

$$8 = d(p_1,\ p_2) \leq d(p_1, x_i) + d(x_i,\ p_2) < 4 + 4 = 8.$$

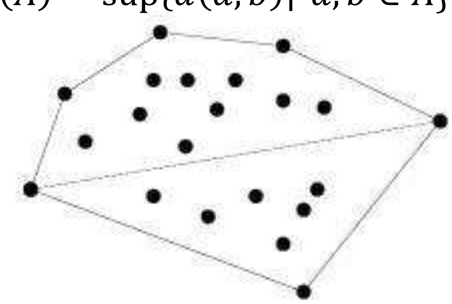
Which is a contradiction ($8 \nless 8$).

Thus each ball $B_4(x_i)$ can contain at most one point of A. Since A has an inifinite number of elements, it can't be covered with a finite number of balls of radius 4.

Hence, A is not totally bounded.

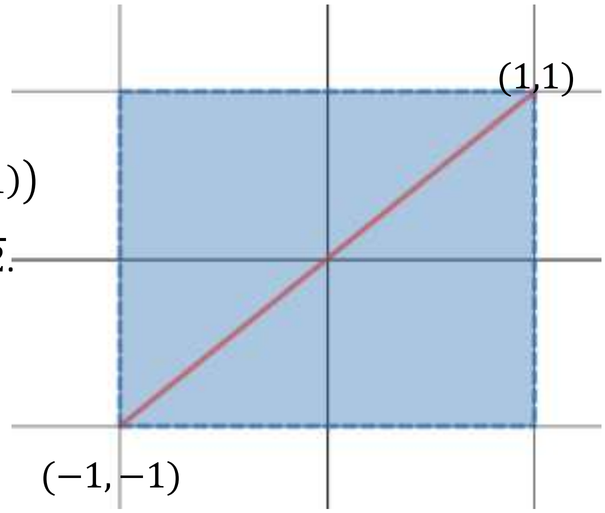
Def. The **diameter** of a subset $A \subseteq M, d$ is:

$$diam(A) = \sup\{d(a, b) |\ a, b \in A\}.$$



Ex. Find the diameter of the set $A = \{(x,y) \in \mathbb{R}^2 |\ |x| < 1,\ |y| < 1\}$.

$$diam(A) = d\big((1,1),(-1,-1)\big)$$
$$= \sqrt{2^2+2^2} = 2\sqrt{2}.$$



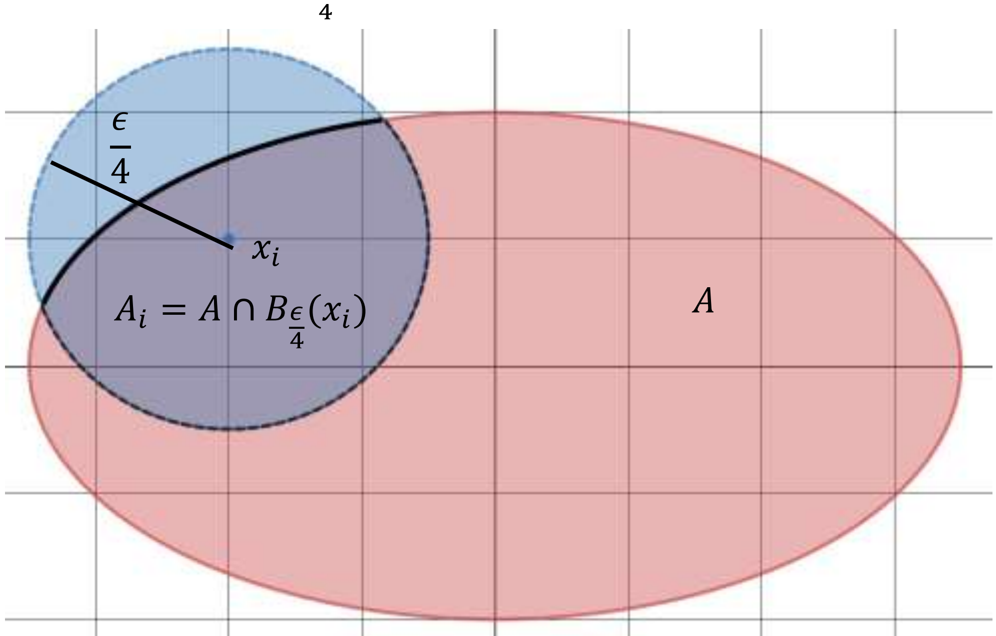
Lemma: A is totally bounded if and only if, given $\epsilon > 0$, there are finitely many sets $A_1, A_2, \dots, A_n \subseteq A$, with $diam(A_i) < \epsilon$ for all $i = 1, \dots, n$, such that $A \subseteq \bigcup_{i=1}^{n} A_i$.

Proof: Assume that A is totally bounded.

Then given $\epsilon > 0$ there exist points $x_1, x_2, \dots, x_n$ such that

$$A \subseteq \bigcup_{i=1}^{n} B_{\frac{\epsilon}{4}}(x_i).$$

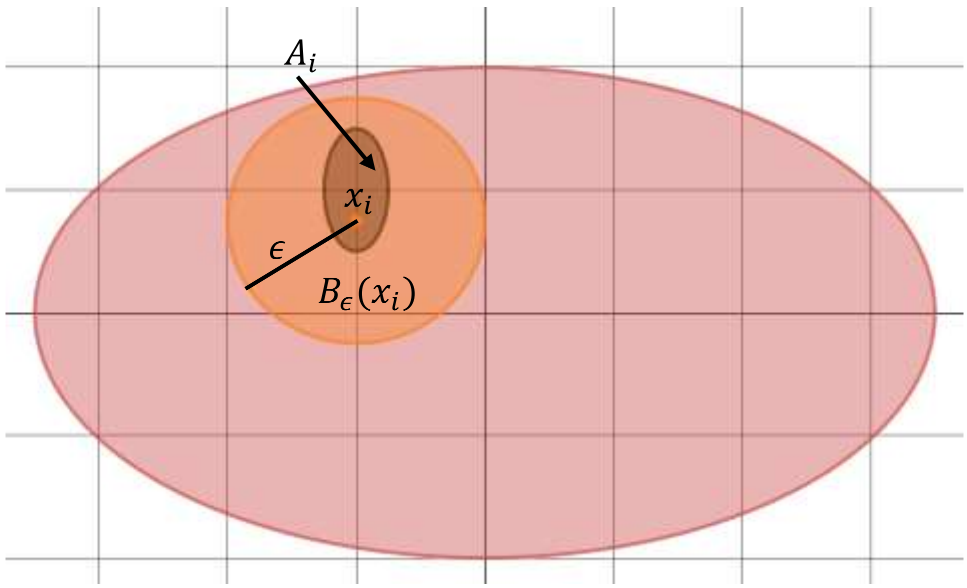
Now let $A_i = A \cap B_{\frac{\epsilon}{4}}(x_i) \subseteq A$.



$$diam(A_i) \le diam\left(B_{\frac{\epsilon}{4}}(x_i)\right) = \frac{\epsilon}{2} < \epsilon \text{ and } A \subseteq \bigcup_{i=1}^{n} A_i.$$

Now assume given any $\epsilon > 0$, there are finitely many sets $A_1, A_2, \ldots, A_n \subseteq A$, with $diam(A_i) < \epsilon$ for all $i = 1, \ldots, n$, such that $A \subseteq \bigcup_{i=1}^{n} A_i$.

Choose any point $x_i \in A_i$. Then $B_\epsilon(x_i) \supseteq A_i$, since $diam(A_i) < \epsilon$.



Thus $A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$ and A is totally bounded.

Def. Recall that a sequence $\{x_n\}$ in a metric space M **converges** to a point $x \in M$ if given any $\epsilon > 0$ there exists a $N \in \mathbb{Z}^+$ such that if $n \ge N$ then $d(x, x_n) < \epsilon$.

Def. Also recall that a sequence $\{x_n\}$ in a metric space M is called **Cauchy** if given any $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ such that if $n, m \ge N$ then $d(x_m, x_n) < \epsilon$.

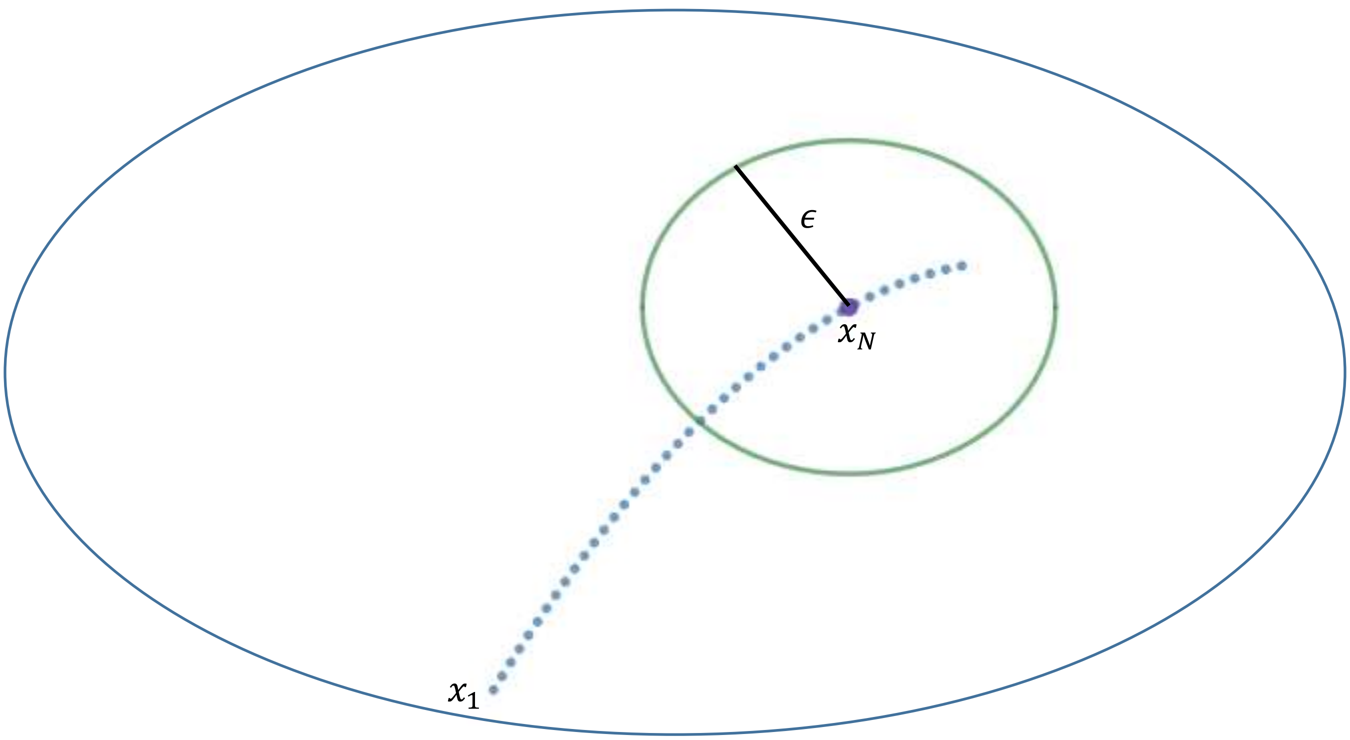
Lemma: Let $\{x_n\}$ be a sequence in a metric space M, and let $A = \{x_n\}$.

i. If $\{x_n\}$ is Cauchy, then A is totally bounded.
ii. If A is totally bounded then $\{x_n\}$ has a subsequence which is Cauchy.

Ex. Let $x_n = (-1)^n$ in $\mathbb{R}$. So in this case $A = \{-1,1\}$, which is totally bounded in $\mathbb{R}$, but the sequence $\{-1,1,-1,1,-1,1,\ldots\}$ does not converge in $\mathbb{R}$. However, the subsequences $\{1,1,1,1\ldots\}$ and $\{-1,-1,-1,\ldots\}$ do converge in $\mathbb{R}$ and hence must be Cauchy.

Proof of Lemma: i. Let $\epsilon > 0$. Since $\{x_n\}$ is a Cauchy sequence there exists an $N \in \mathbb{Z}^+$ such that if $n, m \geq N$ then $d(x_m, x_n) < \epsilon$.

In other words, $diam(\{x_n\}, n \geq N) \leq \epsilon$.



Thus: $A = x_1 \cup x_2 \cup x_3 \cup \ldots \cup x_{N-1} \cup \{x_n | \, n \geq N\}$.

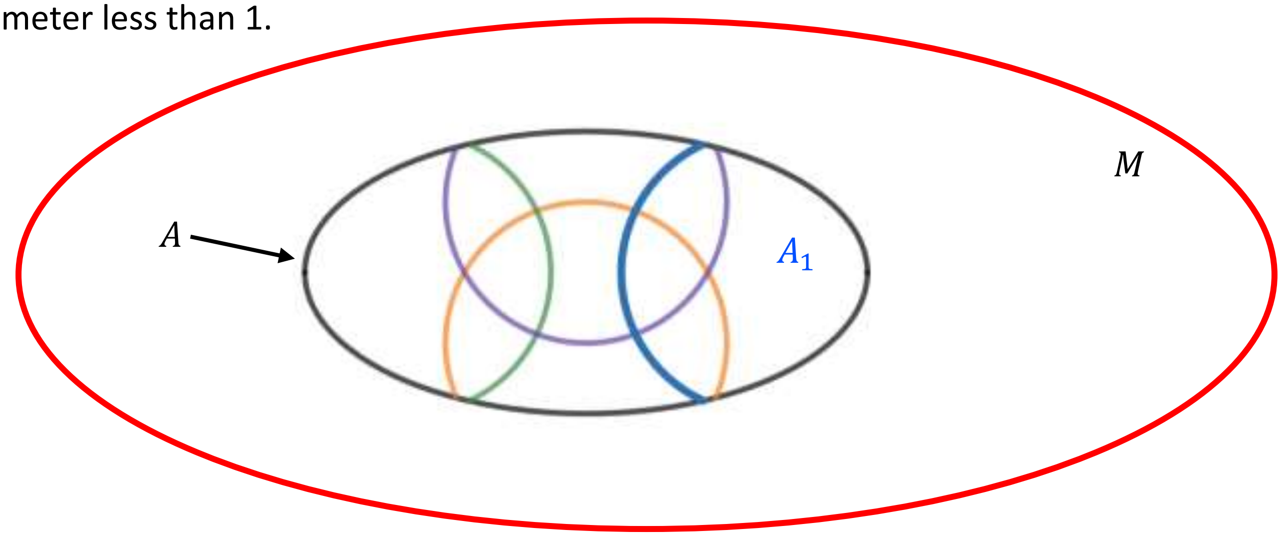
For $i = 1, \ldots, N-1$; $x_i \in B_\epsilon(x_i)$, and $\{x_n | \, n \geq N\} \subseteq B_\epsilon(x_N)$.

Thus $(\bigcup_{i=1}^{N-1} B_\epsilon(x_i)) \cup B_\epsilon(x_N) \supseteq A$, and A is totally bounded..

Now let's assume that A is totally bounded and show that it has a Cauchy subsequence.

If A is a finite set then any sequence must repeat some element of A an infinite number of times and thus has a subsequence which is Cauchy (since it's a constant sequence).

If A is infinite, since it is totally bounded it can be covered by finitely many sets of diameter less than 1.



One of these sets must contain an infinite number of points of A. Call that set A_1.

$A_1 \subseteq A$ so it's totally bounded and can be covered by a finite number of sets of diameter less than $\frac{1}{2}$.

One of these sets must contain an infinite number of points of A_1. Call it A_2.

Continuing the process we get a decreasing sequence of sets:

$$A \supseteq A_1 \supseteq A_2 \supseteq \cdots$$

where A_k contains infinitely many points of A and $diam(A_k) < \frac{1}{k}$.

Choose any element $x_{n_k} \in A_k$, then we have:

$$diam\left(x_{n_j}\middle|\ j \geq k\right) \leq diam(A_k) < \frac{1}{k}.$$

So $\{x_{n_k}\}$ is a Cauchy sequence.

Theorem: A set A is totally bounded if and only if every sequence in A has a Cauchy subsequence.

Proof. The previous lemma showed if A is totally bounded then every sequence in A has a Cauchy subsequence.

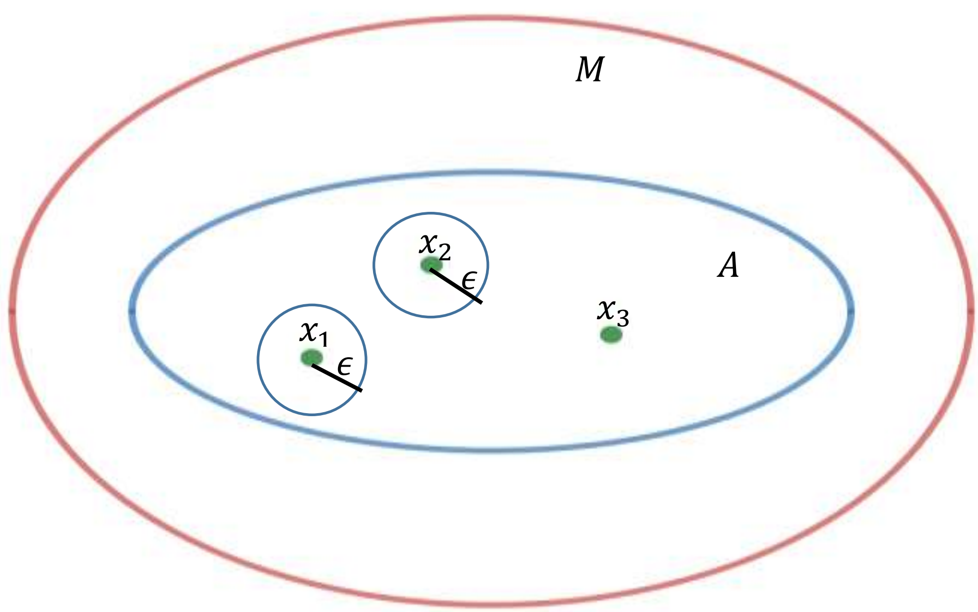
Now let's show if every sequence in A has a Cauchy subsequence then A is totally bounded.

We do this by contradiction. Assume A is not totally bounded.

Then there is some $\epsilon > 0$ such that A cannot be covered by finitely many balls of radius ϵ.

Start with any point $x_1 \in A$.

We can always find a point $x_2 \in A$, with $d(x_1, x_2) \geq \epsilon$.



Given x_2 we can always find $x_3 \in A$, with $d(x_i, x_3) \geq \epsilon,\ i = 1{,}2$.

Continue this process and get a sequence where $d(x_n, x_m) \geq \epsilon,$ for $n \neq m$.

So $\{x_n\}$ has no Cauchy subsequence.

Corollary (The Bolzano-Weierstrass Theorem). Every bounded infinite subset of $\mathbb{R}$ has a limit point in $\mathbb{R}$.

Proof. Let A be a bounded infinite subset of $\mathbb{R}$.

Since A is infinite we can form a sequence $\{x_n\}$ of distinct points in A.

Since A is bounded in $\mathbb{R}$, it is totally bounded.

A is totally bounded so $\{x_n\}$ has a Cauchy subsequence $\{x_{n_k}\}$.

But Cauchy sequences in $\mathbb{R}$ converge since $\mathbb{R}$ is complete, so $x_{n_k} \to x \in \mathbb{R}$.

Thus x is a limit point of A.

Another way to state the Bolzano-Weierstrass theorem is to say that any bounded sequence of real numbers has a convergent subsequence.