Totally Bounded Sets- HW Problems

1. Let M, d be a metric space with $A \subseteq B \subseteq M$. Suppose that B is totally bounded. Prove that A is totally bounded.

2. Suppose $A, B \subseteq M, d$ are totally bounded subsets of M. Prove that $A \cup B$ is totally bounded.

3. Let $M = \{sequences\ of\ real\ numbers\ \{x_i\}\ with\ \sup_{1\leq i<\infty} |x_i| < \infty\}$. Define $d(\{x_i\},\{y_i\}) = \sup_{1\leq i<\infty} |x_i - y_i|$. M is a metric space (called l_∞) with this metric. Let $p \in M$ given by the sequence $p = \{0,0,0,\ldots\}$, the zero sequence.

Let $A = \{\{x_i\} \in M \big| \ d(\{x_i\}, p) \leq 1\}$.

a. Give an example of an element of A (i.e. a sequence $\{x_i\} \in M$ such that $d(\{x_i\}, p) \leq 1$).
b. Prove that A is bounded but not totally bounded.

4. Prove that any bounded subset of $\mathbb{R}$ with the usual metric is totally bounded.

5. Let $\mathbb{R}$ be a metric space with the metric

$$d(x,y) = 1 \quad \text{if } x \neq y$$

$$d(x,y) = 0 \quad \text{if } x = y.$$

Prove that $[0,1] \subseteq \mathbb{R}$ is bounded but not totally bounded.

6. Let $A \subseteq X, d$ a metric space. Suppose there are an infinite number of elements in A , $e_1, e_2, e_3, \ldots \in A$, such that

$$d(e_i, e_j) = 4 \quad \text{if } i \neq j$$

$$d(e_i, e_j) = 0 \quad \text{if } i = j$$

for $i, j = 1,2,3 \ldots$. Prove that A is not totally bounded.