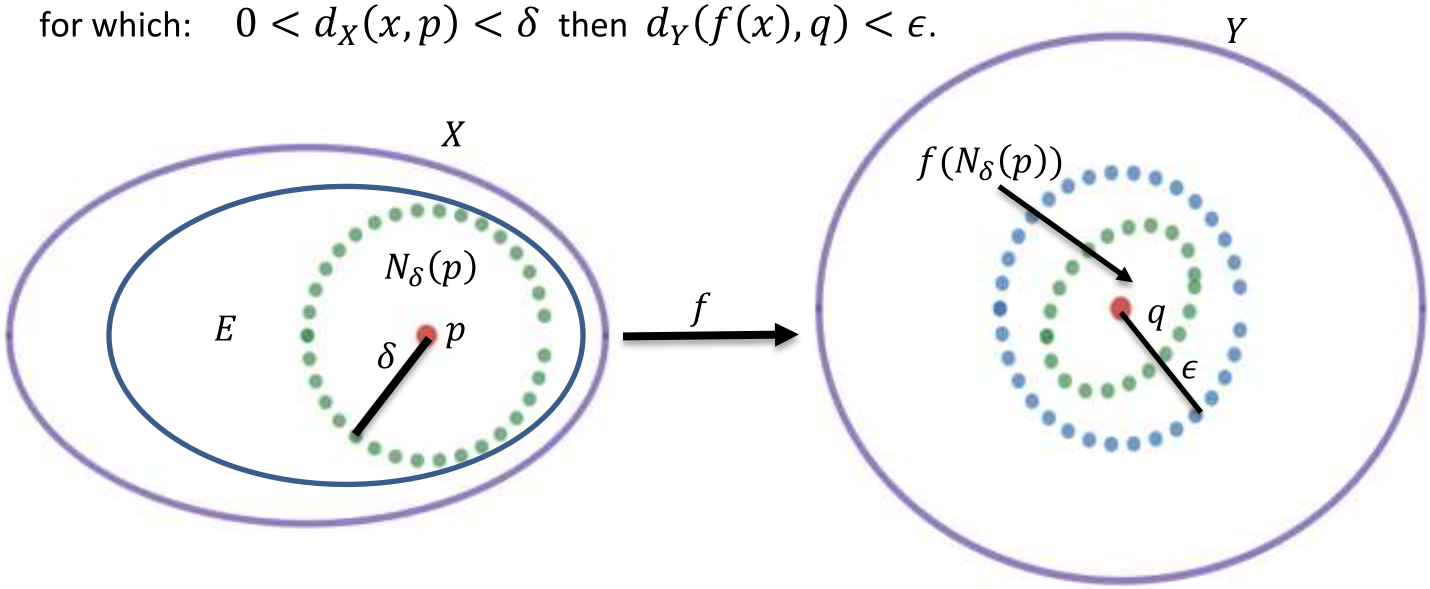
Limits of Functions

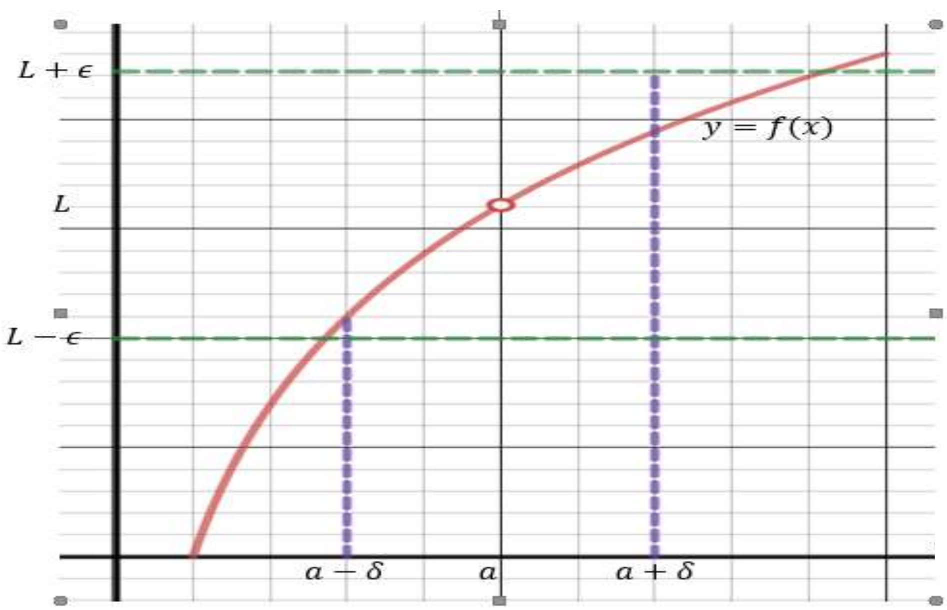
Def. Let X and Y be metric spaces; suppose $E \subseteq X$, $f: E \to Y$, and p is a limit point of E. We write: $\boldsymbol{f(x) \to q}$ **as** $\boldsymbol{x \to p}$ **or** $\boldsymbol{\lim_{x \to p} f(x) = q}$, if there is a point $q \epsilon Y$ such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if for all $x \epsilon E$ for which: $0 < d_X(x,p) < \delta$ then $d_Y(f(x), q) < \epsilon$.



For $X = Y = \mathbb{R}$ this definition says that $\boldsymbol{f(x) \to L}$ **as** $\boldsymbol{x \to a}$ **or**

$\boldsymbol{\lim_{x \to a} f(x) = L}$, means that for all $\epsilon > 0$ there exists a $\delta > 0$ such that if

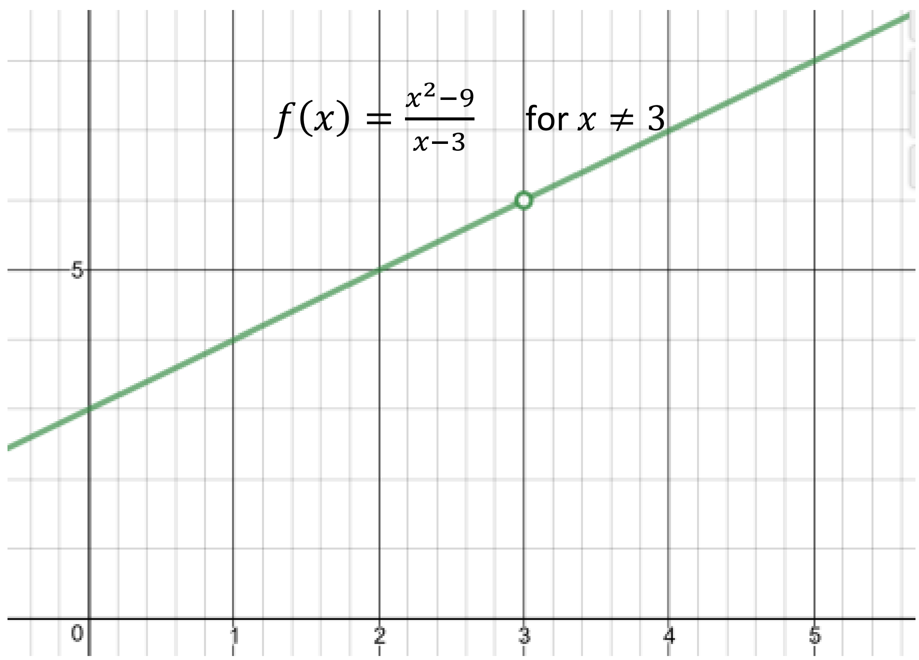
$0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.



Notice that the definition of a limit of a function as x goes to p does NOT depend on the value of the function at $x = p$. In fact, a function can have a limit as x goes to p without the function even being defined at $x = p$.

To prove that $\lim_{x \to p} f(x) = q$, we are going to need to show we can find a $\delta > 0$ that satisfies the conditions in the definition of $\lim_{x \to p} f(x) = q$,. In general, δ will depend on the value of ϵ (i.e., δ will be a function of ϵ) and will depend on the point p.

Ex. Let $f(x) = \frac{x^2-9}{x-3}$ for $x \neq 3$ ($f(x)$ is not defined at $x = 3$). Prove that $\lim_{x \to 3} f(x) = 6$.



By the definition of a limit given earlier, we must show that given any $\epsilon > 0$, we can find a $\delta > 0$ such that if $0 < |x - 3| < \delta$ then $|f(x) - 6| < \epsilon$.

As with proving the limit of a sequence, we start with the ϵ statement and work backwards to see what δ will work. Here we want to get the δ statement to appear.

$$\left|\frac{x^2-9}{x-3}-6\right| = \left|\frac{(x-3)(x+3)}{x-3}-6\right| = |x+3-6| = |x-3| < \epsilon.$$

But $|x-3|$ is exactly what δ controls, i.e., $0 < |x-3| < \delta$.

So just let $\delta = \epsilon$.

Now let's see that this works.

$0 < |x-3| < \delta$ means that since $\delta = \epsilon$

$|x-3| < \epsilon$ (we now work our algebra in reverse to get the ϵ statement)

$$\left|\frac{x^2-9}{x-3}-6\right| = \left|\frac{(x-3)(x+3)}{x-3}-6\right| = |x+3-6| = |x-3| < \epsilon.$$

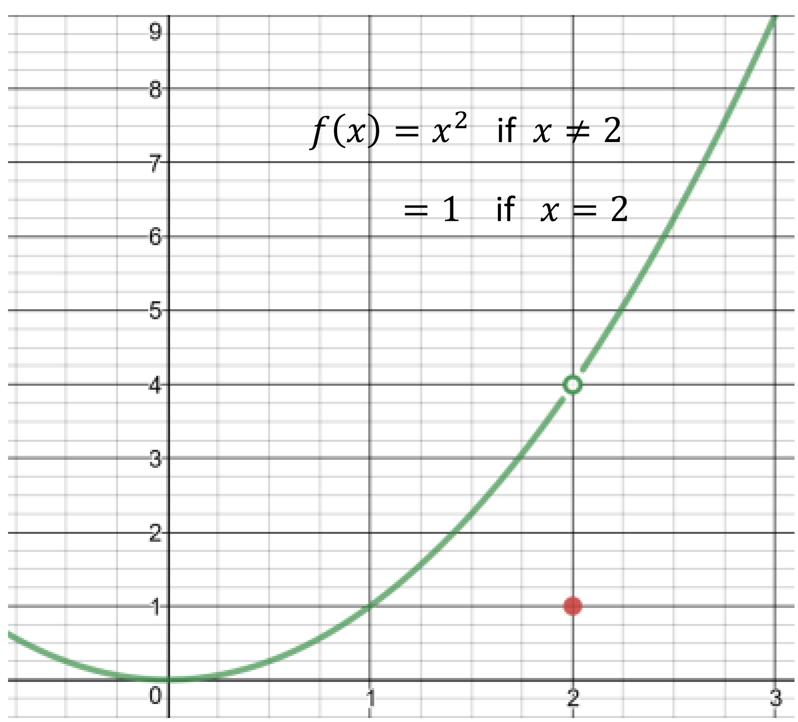
So if $0 < |x-3| < \delta$ then $\left|\frac{x^2-9}{x-3}-6\right| < \epsilon$.

Thus we have proved: $\lim_{x\to 3} f(x) = 6$.

Ex. Let $f(x) = x^2$ if $x \neq 2$

$= 1$ if $x = 2$

Prove $\lim_{x\to 2} f(x) = 4$.



We must show that given any $\epsilon > 0$, we can find a $\delta > 0$ such that if

$0 < |x-2| < \delta$ then $|f(x) - 4| < \epsilon$.

Let's start with the ϵ statement and work backwards until the δ statement appears.

For $x \neq 2$, $f(x) = x^2$. So the ϵ statement is:

$|x^2 - 4| < \epsilon$

$|x^2 - 4| = |(x+2)(x-2)| = |x+2||x-2|$.

$|x-2|$ is part of the δ statement, but what do we do about the factor $|x+2|$?

Notice that if $\delta \leq 1$, ie, $1 < x < 3$, then $3 < x + 2 < 5$ or $|x+2| < 5$.

So if $\delta \leq 1$, then $|x^2 - 4| = |x+2||x-2| < 5|x-2|$.

Thus, If we can ensure that $5|x-2| < \epsilon$, then $|x^2 - 4| < \epsilon$.

Equivalently, if we can ensure that $|x-2| < \frac{\epsilon}{5}$ then $|x^2 - 4| < \epsilon$.

So choose $\delta = \min(1, \frac{\epsilon}{5})$.

Let's show that this δ works, i.e., that if $0 < |x-2| < \delta$ then $|x^2 - 4| < \epsilon$.

$0 < |x-2| < \delta = \min(1, \frac{\epsilon}{5})$

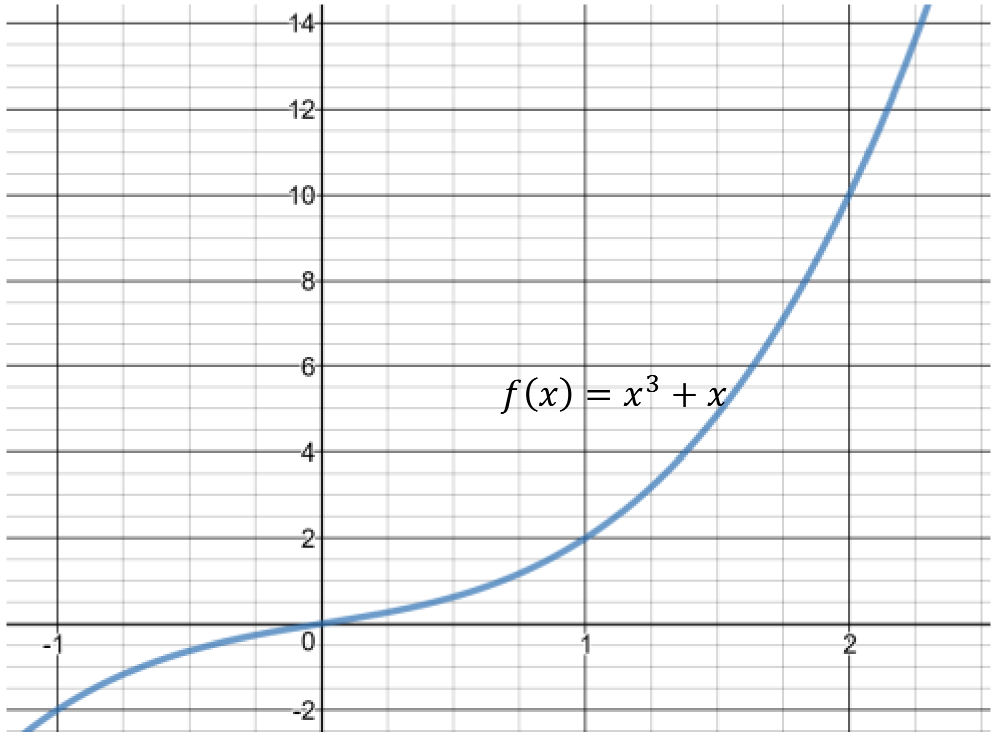
$|x^2 - 4| = |x+2||x-2|$; since $\delta \leq 1$ we know that $|x+2| < 5$, so

$|x^2 - 4| = |x+2||x-2| < 5|x-2|$; since $\delta \leq \frac{\epsilon}{5}$ we have:

$|x^2 - 4| = |x+2||x-2| < 5|x-2| < 5\delta \leq 5\left(\frac{\epsilon}{5}\right) = \epsilon$.

So we have proved that $\lim_{x \to 2} f(x) = 4$.

Ex. Let $f(x) = x^3 + x$, prove $\lim_{x \to 2} f(x) = 10$.



We must show that given any $\epsilon > 0$, we can find a $\delta > 0$ such that if

$0 < |x - 2| < \delta$ then $|x^3 + x - 10| < \epsilon$.

By dividing $(x - 2)$ into $x^3 + x - 10$ we get

$x^3 + x - 10 = (x - 2)(x^2 + 2x + 5)$ so we have:

$|x^3 + x - 10| = |x - 2||x^2 + 2x + 5|$.

So, once again, we have the δ statement popping out. The problem is, what do we do about $|x^2 + 2x + 5|$? We again use the trick of limiting $\delta \leq 1$ and ask how big $|x^2 + 2x + 5|$ could possibly be?

Since $\delta \leq 1$, ie, $1 < x < 3$, we know $|x| < 3$. By the triangle inequality:

$$|x^2 + 2x + 5| \leq |x^2| + 2|x| + 5 < 9 + 6 + 5 = 20.$$

So we have:

$$|x^3 + x - 10| = |x - 2||x^2 + 2x + 5| < 20|x - 2|.$$

Now if we can choose a δ such that $|x^3 + x - 10| < 20|x - 2| < \epsilon$

We'll be effectively done. But this is equivalent to: $|x - 2| < \frac{\epsilon}{20}$.

So let's choose $\delta = \min(1, \frac{\epsilon}{20})$.

Let's show that this delta works (if $0 < |x - 2| < \delta$ then $|x^3 + x - 10| < \epsilon$).

If $\delta = \min(1, \frac{\epsilon}{20})$ we have:

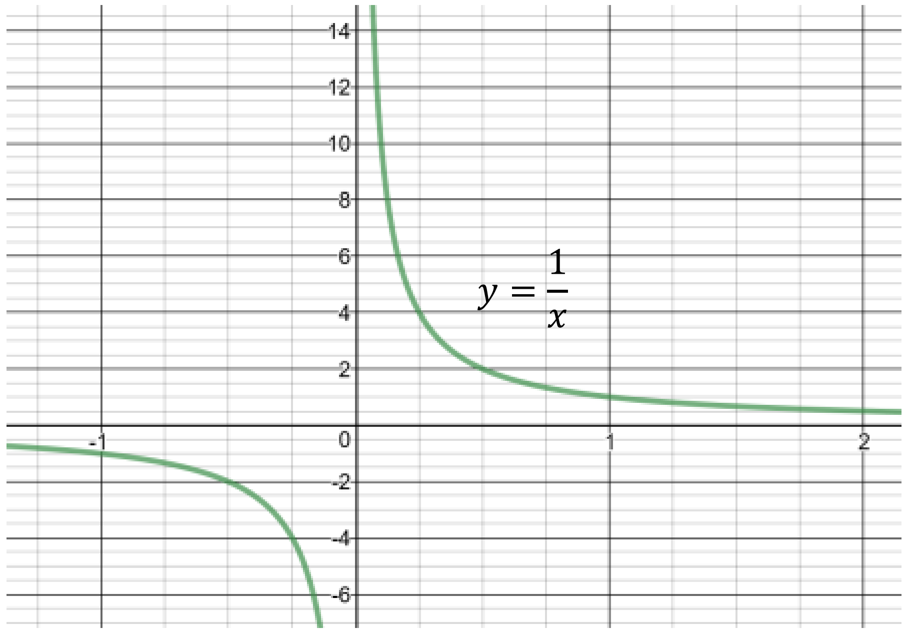
$|x^3 + x - 10| = |x - 2||x^2 + 2x + 5|$; but since $\delta \leq 1$ we know that:

$|x^3 + x - 10| = |x - 2||x^2 + 2x + 5| < 20|x - 2|$; Since $\delta \leq \frac{\epsilon}{20}$:

$|x^3 + x - 10| < 20|x - 2| < 20\delta \leq 20\left(\frac{\epsilon}{20}\right) = \epsilon.$

So we have proved that $\lim_{x \to 2} f(x) = 10.$

Ex. Prove $\lim_{x \to 1} \frac{1}{x} = 1$.



We must show that given any $\epsilon > 0$, we can find a $\delta > 0$ such that if

$0 < |x - 1| < \delta$ then $\left|\frac{1}{x} - 1\right| < \epsilon$.

We start with the ϵ statement and work backwards.

$$\left|\frac{1}{x} - 1\right| = \left|\frac{1}{x} - \frac{x}{x}\right| = \left|\frac{1-x}{x}\right| = \left|\frac{1}{x}\right| |x - 1| .$$

Now we need an upper bound on $|\frac{1}{x}|$.

NOTICE, we can't just let $\delta \leq 1$. Because if we let $\delta = 1$ then

$0 < |x - 1| < \delta = 1$, means x can be VERY close to 0 and hence

$\frac{1}{x}$ won't have an upper bound.

However, there's nothing magical about letting $\delta \leq 1$ (it just tends to be easy to work with). We just want to make sure x stays away from 0, so choose

$\delta \leq \frac{1}{2}$ (or any number less than 1 and greater than 0).

This means that: $|x-1| < \frac{1}{2}$

$-\frac{1}{2} < x - 1 < \frac{1}{2}$ (now add 1 to all quantities)

$\frac{1}{2} < x < \frac{3}{2}$ (now take reciprocals since all terms have same sign)

$$2 > \frac{1}{x} > \frac{2}{3}$$

So: $\left|\frac{1}{x}\right| < 2$ if $\delta \leq \frac{1}{2}$.

So if $\delta \leq \frac{1}{2}$ we have:

$$\left|\frac{1}{x} - 1\right| = \left|\frac{1-x}{x}\right| = \left|\frac{1}{x}\right| |x-1| < 2|x-1| < \epsilon.$$

$2|x-1| < \epsilon$ is equivalent to $|x-1| < \frac{\epsilon}{2}$.

So choose $\delta = min(\frac{1}{2}, \frac{\epsilon}{2})$.

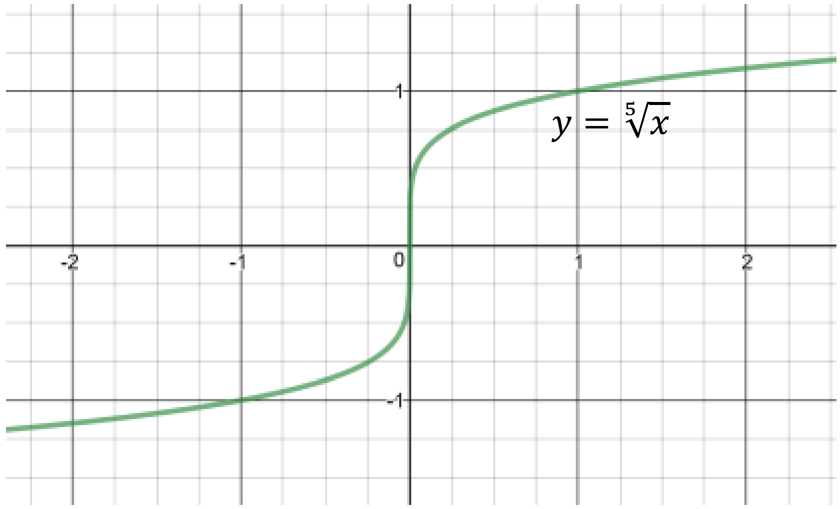
Now let's show this δ works:

$$\left|\frac{1}{x} - 1\right| = \left|\frac{1}{x} - \frac{x}{x}\right| = \left|\frac{1-x}{x}\right| = \left|\frac{1}{x}\right| |x-1| < 2|x-1| \quad \text{Since } \delta \leq \frac{1}{2}.$$

$$< 2\delta \leq 2\left(\frac{\epsilon}{2}\right) = \epsilon \quad \text{Since } \delta \leq \frac{\epsilon}{2}.$$

So $\lim_{x \to 1} \frac{1}{x} = 1$.

Ex. Prove $\lim_{x \to 0} \sqrt[5]{x} = 0.$



We must show that given any $\epsilon > 0$, we can find a $\delta > 0$ such that if

$0 < |x - 0| < \delta$ then $\left|\sqrt[5]{x} - 0\right| < \epsilon.$

Or, equivalently, if $0 < |x| < \delta$ then $\left|\sqrt[5]{x}\right| < \epsilon.$

So we need: $\left|\sqrt[5]{x}\right| = \sqrt[5]{|x|} < \epsilon$ or $|x| < \epsilon^5.$

Choose $\delta = \epsilon^5.$

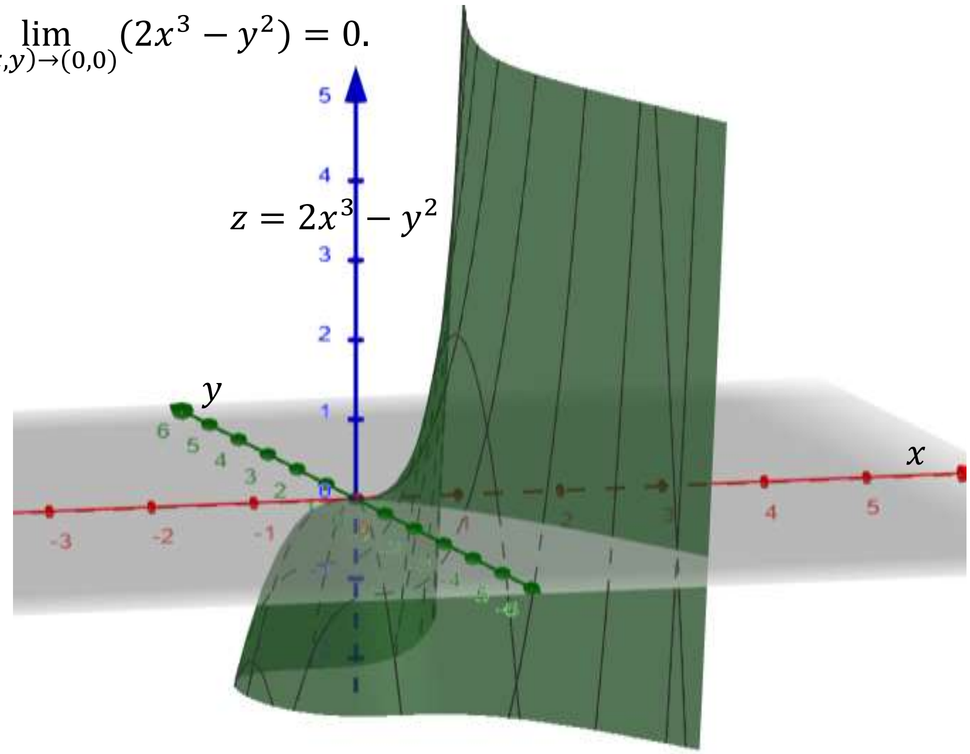
Now let's show that this δ works:

$0 < |x| < \delta$ means that $|x| < \delta = \epsilon^5$ or equivalently: $\left|\sqrt[5]{x}\right| < \epsilon$

This is algebraically the same as: $\left|\sqrt[5]{x} - 0\right| < \epsilon.$

So we have shown that $\lim_{x \to 0} \sqrt[5]{x} = 0.$

Ex. Prove that $\lim_{(x,y)\to(0,0)} (2x^3 - y^2) = 0.$



We must show that given any $\epsilon > 0$, we can find a $\delta > 0$ such that if

$0 < d\big((x,y),(0,0)\big) < \delta$ then $|2x^3 - y^2 - 0| < \epsilon$.

i.e. if $0 < \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2} < \delta$ then $|2x^3 - y^2| < \epsilon$.

Let's start with the ϵ statement and work backards toward the δ statement.

Using the triangle inequality we have:

$|2x^3 - y^2| \le |2x^3| + |y^2| = 2|x|^3 + |y|^2$

Since $\sqrt{x^2 + y^2} < \delta$ we have $|x| < \sqrt{x^2 + y^2} < \delta$ and $|y| < \sqrt{x^2 + y^2} < \delta$.

Now choose $\delta \le 1$, so $|x| < \delta \le 1$ and $|y| < \delta \le 1$.

Notice that $|x|^3 < |x|$ and $|y|^2 < |y|$ since $|x| < 1$ and $|y| < 1$, So we have:

$$|2x^3 - y^2| \le 2|x|^3 + |y|^2 < 2|x| + |y| < 2\delta + \delta = 3\delta.$$

So we need to force

$$|2x^3 - y^2| < 3\delta < \epsilon.$$

Or $\delta < \frac{\epsilon}{3}$.

Choose $\delta = \min(1, \frac{\epsilon}{3})$.

Now let's show that this δ works.

If $0 < \sqrt{x^2 + y^2} < \delta$ then:

$$|2x^3 - y^2| \le |2x^3| + |y^2| = 2|x|^3 + |y|^2 \text{ ; so}$$

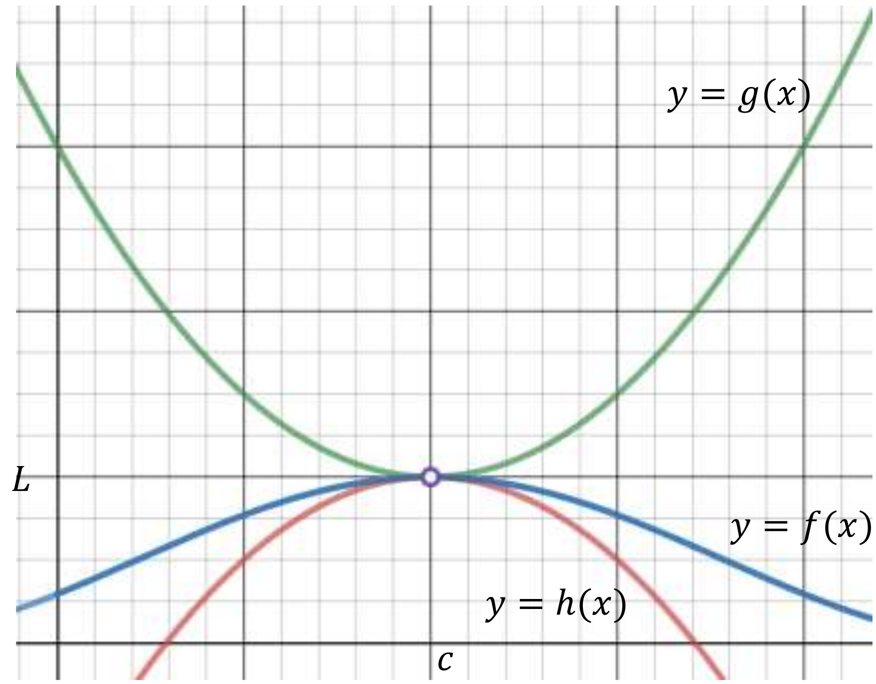
$$|2x^3 - y^2 - 0| \le 2|x|^3 + |y|^2 < 2|x| + |y| \qquad \text{because } \delta \le 1.$$

Since $\sqrt{x^2 + y^2} < \delta \Rightarrow |x| < \delta$ and $|y| < \delta$, we have:

$$|2x^3 - y^2 - 0| < 2|x| + |y| < 3\delta \le 3\left(\frac{\epsilon}{3}\right) = \epsilon \text{ . } \quad \text{because } \delta \le \frac{\epsilon}{3}.$$

Thus we have shown that $\lim_{(x,y)\to(0,0)} (2x^3 - y^2) = 0$.

Theorem (Squeeze theorem): If $h(x) \le f(x) \le g(x)$ for $x\epsilon(a,b)$, except possibly at $c\epsilon(a,b)$, and if $\lim_{x\to c} h(x) = \lim_{x\to c} g(x) = L$, then $\lim_{x\to c} f(x)$ exists and equals L.



Proof:

Given any $\epsilon > 0$ we need to show that there exists a $\delta > 0$ such that if

$0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$.

Since $\lim_{x\to c} h(x) = \lim_{x\to c} g(x) = L$ we know that given any $\epsilon > 0$ there exists a $\delta_1 > 0$ and a $\delta_2 > 0$ such that if:

$0 < |x - c| < \delta_1$ then $|h(x) - L| < \epsilon$

$0 < |x - c| < \delta_2$ then $|g(x) - L| < \epsilon$.

Let's let $\delta = \min(\delta_1, \delta_2)$.

Thus if $0 < |x - c| < \delta$ then we know both:

$|h(x) - L| < \epsilon$ and $|g(x) - L| < \epsilon$.

Or equivalently:

$-\epsilon < h(x) - L < \epsilon$ and $-\epsilon < g(x) - L < \epsilon$.

In particular, we know that if $0 < |x - c| < \delta$ then:

$-\epsilon < h(x) - L$ and $g(x) - L < \epsilon$ or

$L - \epsilon < h(x)$ and $g(x) < L + \epsilon$.

But by assumption $h(x) \leq f(x) \leq g(x)$ so we have:

$$L - \epsilon < h(x) \leq f(x) \leq g(x) < L + \epsilon$$

$L - \epsilon < f(x) < L + \epsilon$ which is the same as:

$|f(x) - L| < \epsilon$ if $0 < |x - c| < \delta$

So we have shown that $\lim_{x \to c} f(x) = L$.

Ex. Prove $\lim_{x \to 0} x sin(\frac{1}{x}) = 0$.

Since $|sin(\frac{1}{x})| \leq 1$ we have that $0 \leq |x sin(\frac{1}{x})| \leq |x|$

Now let $h(x) = 0$, $f(x) = |x sin(\frac{1}{x})|$, $g(x) = |x|$

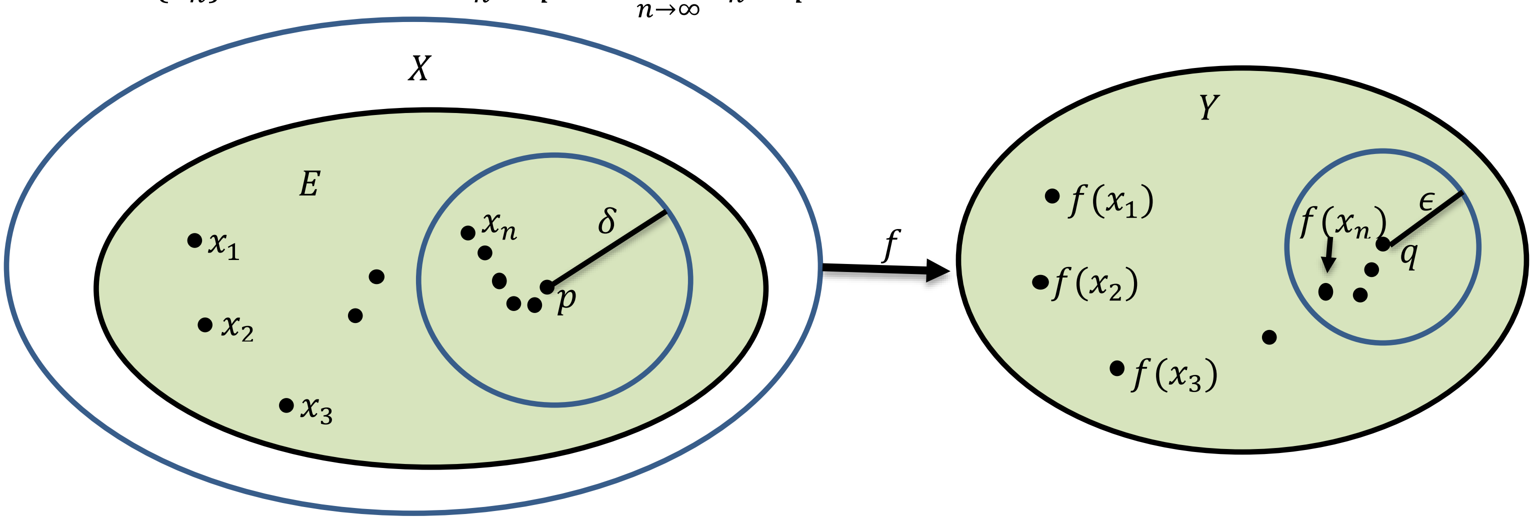
Since $\lim_{x \to 0} |x| = 0$ and $\lim_{x \to 0} 0 = 0$, we know that $\lim_{x \to 0} |x sin\left(\frac{1}{x}\right)| = 0$.

From an earlier HW problem we know that for a sequence $\{a_n\}$, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$. It is also true for limits of functions:

$\lim_{x\to c} f(x) = 0$ if an only if $\lim_{x\to c} |f(x)| = 0$.

Thus $\lim_{x\to 0} x sin\left(\frac{1}{x}\right) = 0$.

Theorem: Let X, Y be metric spaces with $f: E \subseteq X \to Y$, with p a limit point of E and $q \epsilon Y$, then $\lim_{x\to p} f(x) = q$ if and only if $\lim_{n\to\infty} f(x_n) = q$ for every sequence $\{x_n\} \subseteq E$ such that $x_n \neq p$ and $\lim_{n\to\infty} x_n = p$.



Proof: Assume $\lim_{x\to p} f(x) = q$ and we will show that given any sequence $\{x_n\} \subseteq E$ such that $x_n \neq p$ and $\lim_{n\to\infty} x_n = p$, that $\lim_{n\to\infty} f(x_n) = q$.

Let $\epsilon > 0$ be given. By definition of $\lim_{x \to p} f(x) = q$, there exists a $\delta > 0$ such that if $x \epsilon E$ and $0 < d_X(x,p) < \delta$ then $d_Y(f(x), q) < \epsilon$.

Since $\lim_{n \to \infty} x_n = p$, by definition, there exists an N such that if $n \geq N$ then $0 < d_X(x_n, p) < \delta$ (for the above δ).

So for $n \geq N$, $0 < d_X(x_n, p) < \delta$, and $d_Y(f(x_n), q) < \epsilon$

Thus $\lim_{n \to \infty} f(p_n) = q$.

Now we assume that $\lim_{n \to \infty} f(x_n) = q$ for every sequence $\{x_n\} \subseteq E$ such that $x_n \neq p$ and $\lim_{n \to \infty} x_n = p$ and show that $\lim_{x \to p} f(x) = q$.

We will do this with a proof through contradiction.

Let's assume that the conclusion is false, ie that $\lim_{x \to p} f(x) \neq q$.

Then there exists some $\epsilon > 0$ such that for every $\delta > 0$ there exists a point $x \epsilon E$ (that depends on δ) for which $d_Y(f(x), q) \geq \epsilon$, but $0 < d_X(x,p) < \delta$.

Take $\delta_n = \frac{1}{n}$; $n = 1, 2, 3, \ldots$

For each δ_n there exists some point x_n with $d_Y(f(x_n), q) \geq \epsilon$.

But then $\lim_{x \to p} f(x_n) \neq q$ which contradicts our assumption that $\lim_{n \to \infty} f(x_n) = q$.

So $\lim_{x \to p} f(x) = q$.

Def. $f, g: X \to \mathbb{R}$, X a metric space. Then:

1. $(f \pm g)(x) = f(x) \pm g(x)$
2. $(fg)(x) = f(x)g(x)$
3. $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$; $g(x) \neq 0$

$f, g: X \to \mathbb{R}^k$

1. $(f \pm g)(x) = f(x) \pm g(x)$ (vector addition/subtraction)
2. $(f \cdot g)(x) = f(x) \cdot g(x)$ (dot product of vectors)
3. $(\lambda f)(x) = \lambda f(x)$ (scalar multiplication, $\lambda \epsilon \mathbb{R}$).

Theorem: Suppose $E \subseteq X$ a metric space, p a limit point of E, and $f, g: E \to \mathbb{R}$ with $\lim_{x \to p} f(x) = A$ and $\lim_{x \to p} g(x) = B$, then:

a. $\lim_{x \to p} (f \pm g)(x) = A \pm B$

b. $\lim_{x \to p} fg(x) = A\, B$

c. $\lim_{x \to p} \frac{f}{g}(x) = \frac{A}{B}$; if $B \neq 0$.

Proof. All 3 follow from the previous theorem and the analogous theorem for sequences.