Upper and Lower Limits

Def. Let $\{s_n\}$ be a sequence of real numbers such that:

1. If for every real number M there is a positive integer N such that if $n \geq N$ then $s_n \geq M$, then we say $\lim_{n\to\infty} s_n = +\infty$

2. If for every real M there is a positive integer N such that if $n \geq N$ then $s_n \leq M$, then we say $\lim_{n\to\infty} s_n = -\infty$.

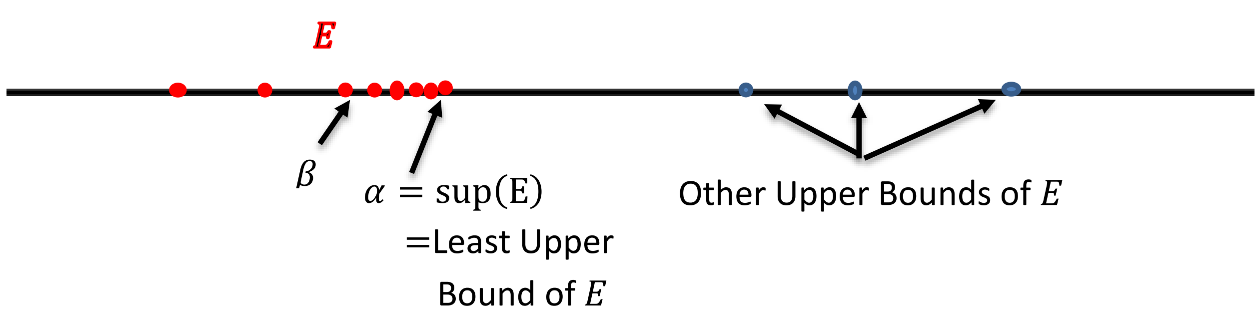
Def. Suppose $E \subseteq \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ and that there exists an $\alpha \epsilon \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ such that:

i. $x \leq \alpha$ for all $x \epsilon E$

ii. if $\beta < \alpha$ then β is not an upper bound for E

then α is called the **Least Upper Bound** for E, or **Supremum** of E, and we write:

$$\alpha = supE.$$

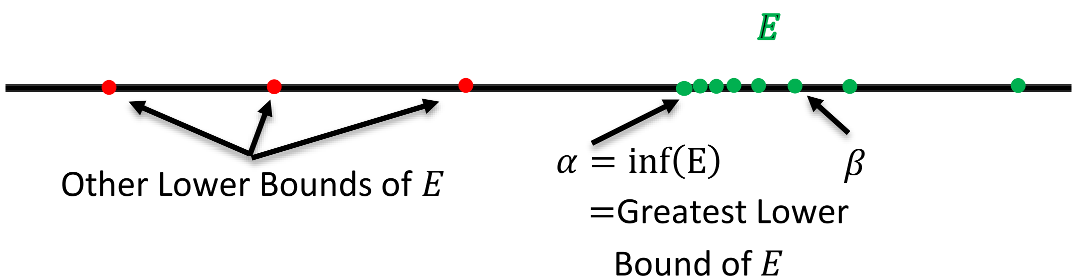


If $\alpha \epsilon \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ such that:

i. $x \geq \alpha$ for all $x \epsilon E$

ii. if $\beta > \alpha$ then β is not an lower bound for E

then we say α is the **Greatest Lower Bound** for E, or the **Infimum** of E, and we write: $\alpha = infE.$



Notice that $infE$ and $supE$ do not have to lie in E.

Ex. Let $E = (0,1)$.

inf $E = 0$ and sup $E = 1$, neither of which lie in E.

Ex. Let $E = [0, \infty)$

$$infE = 0, \quad supE = +\infty$$

Ex. Let $E = \{x \epsilon \mathbb{R} |\ 2 < x^2 < 3\}$

$$infE = -\sqrt{3}, \quad supE = \sqrt{3}$$

Def. Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of

$x \epsilon \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$. This set E contains all subsequential limits of $\{s_n\}$ (including $+\infty$ $and - \infty$, if they are subsequential limits).

$$s^* = supE = \lim_{n\to\infty} \sup(s_n) = \boldsymbol{upper\ limit\ of\ \{s_n\}}$$

$$s_* = infE = \lim_{n\to\infty} inf(s_n) = \boldsymbol{lower\ limit\ of\ \{s_n\}}.$$

Ex. If a sequence $\{s_n\}$ has a limit L, (e.g. $\{\frac{n}{n+1}\} \to 1\}$ then

$$E = \{L\}$$

$\lim_{n\to\infty} \sup(s_n) = \lim_{n\to\infty} inf(s_n) = L,$ i.e., the upper limit=lower limit=L.

Ex. Let $\{s_n\} = \{1, -1, 1, -1, 1, -1, \dots\}$; where $s_{2k-1} = 1,\ \ s_{2k} = -1$

$$E = \{-1, 1\}$$

$s^* = supE = \lim_{n\to\infty} \sup(s_n) = 1$= upper limit of $\{s_n\}$

$s_* = infE = \lim_{n\to\infty} inf(s_n) = -1$ =lower limit of $\{s_n\}$

Ex. Let $\{s_n\}$ =all rational numbers.

Since the rational numbers are dense in the real numbers, every real number is a subsequential limit of $\{s_n\}$ as well as ∞ and $-\infty$. Thus we have:

$$E = \mathbb{R} \cup \{\infty\} \cup \{-\infty)$$

$$s^* = supE = \lim_{n\to\infty} \sup(s_n) = +\infty \text{= upper limit of } \{s_n\}$$

$$s_* = infE = \lim_{n\to\infty} inf(s_n) = -\infty \text{ =lower limit of } \{s_n\}.$$

Ex. Let $\{s_n\}$ be defined by: $s_{2n} = (-1)^n(\frac{n}{2(n+1)})$, $\qquad s_{2n-1} = \frac{2n}{n+1}$

$$\{s_n\} = \{1, \frac{-1}{4}, \frac{4}{3}, \frac{1}{3}, \frac{6}{4}, \frac{-3}{8}, \frac{8}{5}, \frac{4}{10}, \ldots\}$$

$$E = \{\frac{-1}{2}, \frac{1}{2}, 2\}$$

$$s^* = supE = \lim_{n\to\infty} \sup(s_n) = 2 \text{= upper limit of } \{s_n\}$$

$$s_* = infE = \lim_{n\to\infty} inf(s_n) = -\frac{1}{2} \text{ =lower limit of } \{s_n\}.$$