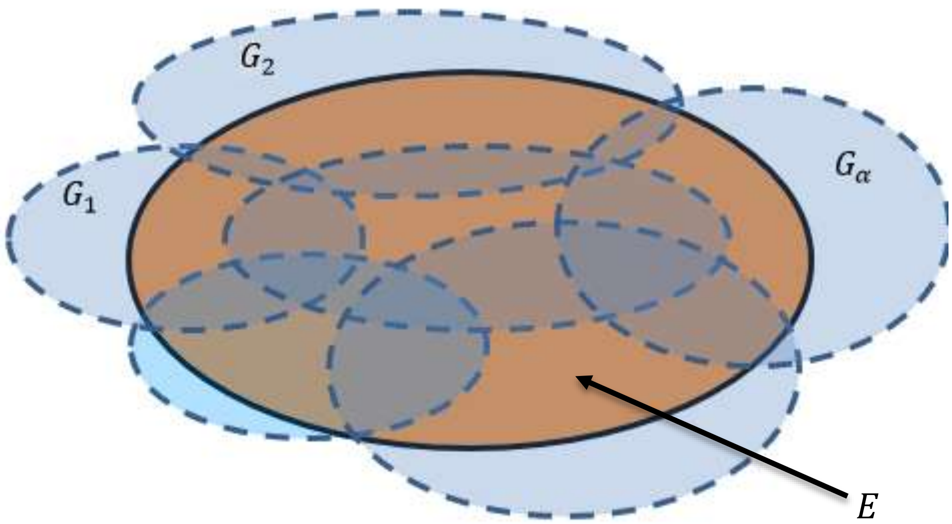
Compact Sets and Connected Sets

Compact Sets

Def. By an **open cover** of a set $E \subseteq X$, a metric space, we mean a collection $\{G_\alpha\}$ of open sets in X such that $E \subseteq \bigcup_\alpha G_\alpha$.



Open Cover of E

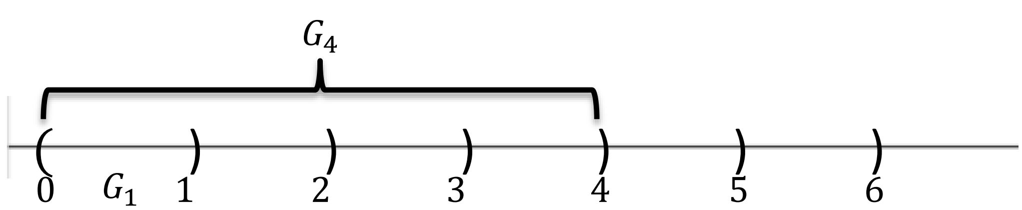
Ex. Let $G_i = (0, i) \subseteq \mathbb{R}$. Then $\{G_i\}_{i=1}^{\infty}$ is an open cover of $(0, \infty)$ (it's also an open cover of $(0, n)$, $[1,7]$, $etc.$)

$G_1 = (0,1)$

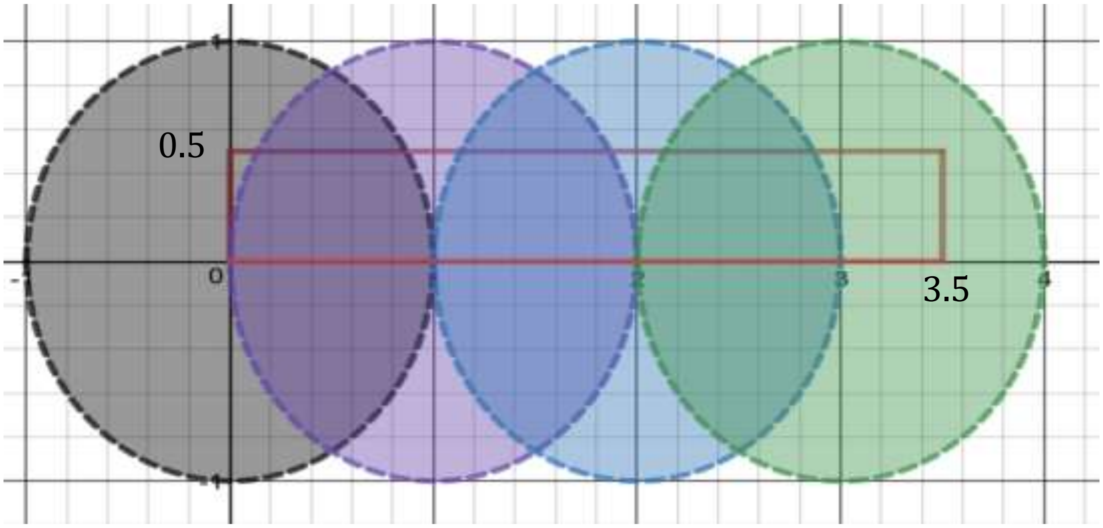
$G_2 = (0,2)$

$G_3 = (0,3)$

$\vdots$

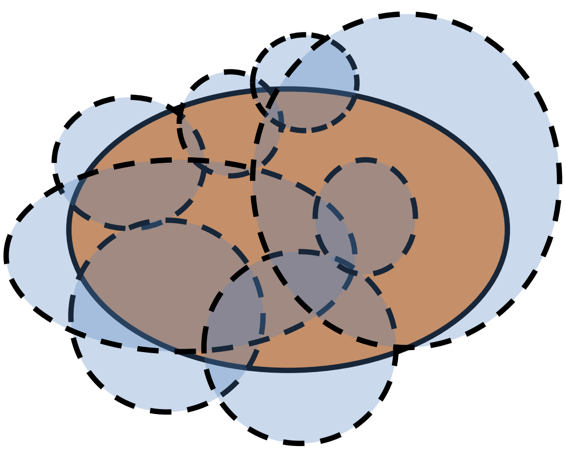


Ex. Let $G_i = \{(x, y) | (x - i)^2 + y^2 < 1\}$, $i = 0,1,2,3$. $\{G_i\}_{i=0}^{3}$ is an open cover of $R = \{(x, y) | 0 \leq x \leq 3.5,\ 0 \leq y \leq \frac{1}{2}\}$.

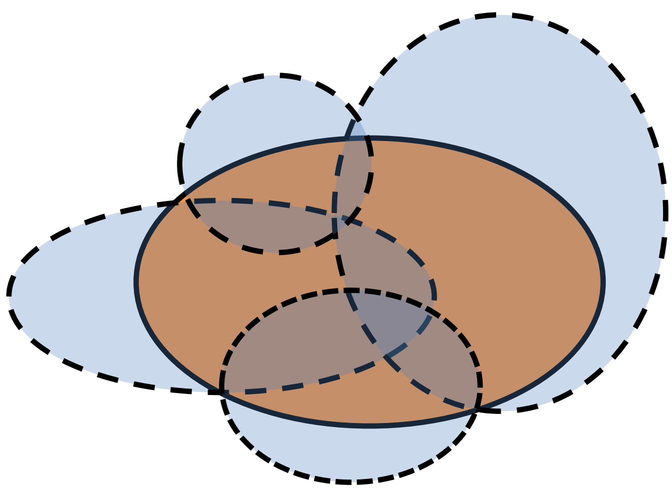


Def. A subset $K \subseteq X, d$ is said to be **compact** if every open cover contains a finite subcover.

This means that if $\{G_\alpha\}$ is any open cover of K, i.e., $K \subseteq \bigcup_\alpha G_\alpha$, then there exist $G_{i_1}, G_{i_2}, \ldots, G_{i_n}$ such that $K \subseteq G_{i_1} \cup G_{i_2} \cup \ldots \cup G_{i_n}$.



Open Cover of K



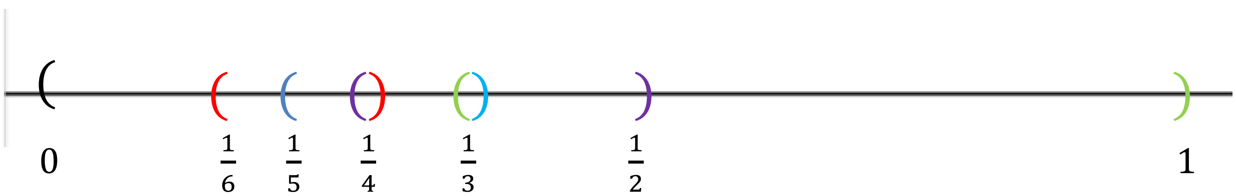
Finite Subcover of K

Ex. Let $K = (0,1) \subseteq \mathbb{R}$. Show K is not compact.

To show a set is not compact we just need to show that we can find an open cover that does not have a finite subcover.

Let $G_i = (\frac{1}{i+2}, \frac{1}{i})$. So we have:

$G_1 = (\frac{1}{3}, 1)$, $G_2 = (\frac{1}{4}, \frac{1}{2})$, $G_3 = (\frac{1}{5}, \frac{1}{3})$, $G_4 = (\frac{1}{6}, \frac{1}{4})$, ...



Notice that $\bigcup_{i=1}^{\infty} G_i = (0,1)$, therefore, $\{G_i\}$ is an open cover of $(0,1)$.

Suppose there was a finite subcover of the $\{G_i\}$, made up of the intervals:

$$(a_1, b_1),\ (a_2, b_2),\ (a_3, b_3), \ldots, (a_n, b_n).$$

Let $m = \min(a_1, a_2, \ldots, a_n)$.

Then the number $\frac{m}{2} \notin \bigcup_{i=1}^{n}(a_i, b_i)$, but $\frac{m}{2} \in (0,1)$.

Hence, $(a_1, b_1),\ (a_2, b_2),\ (a_3, b_3), \ldots, (a_n, b_n)$ is not an open cover of $(0,1)$.

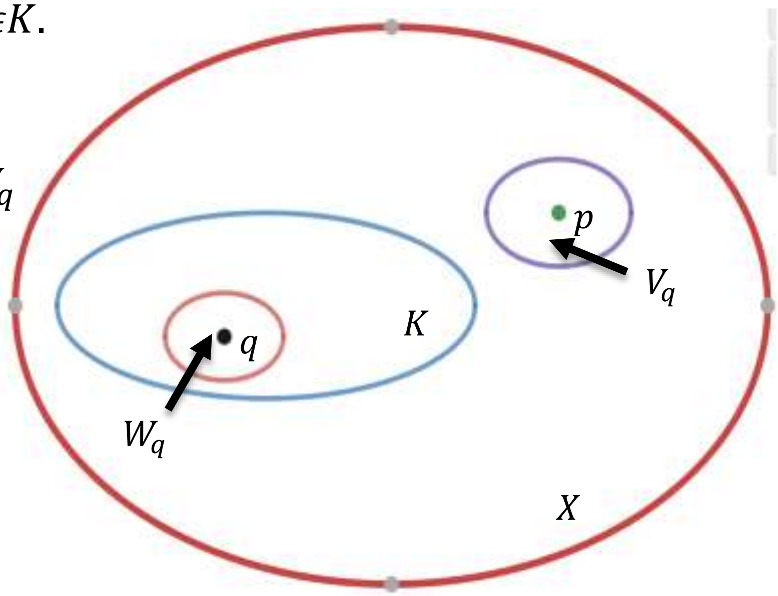
Thus K is not compact.

Theorem: Compact subsets of metric spaces are closed.

Proof: Let $K \subseteq X, d$ be a compact subset of a metric space X.

In order to show that K is closed, we will show that K^c is open.

Suppose $p \epsilon X$, but $p \notin K$, and $q \epsilon K$.



For each point $q \epsilon K$, let V_q and W_q be neighborhoods of p and q with radii less than $\frac{1}{2}d(p,q)$.

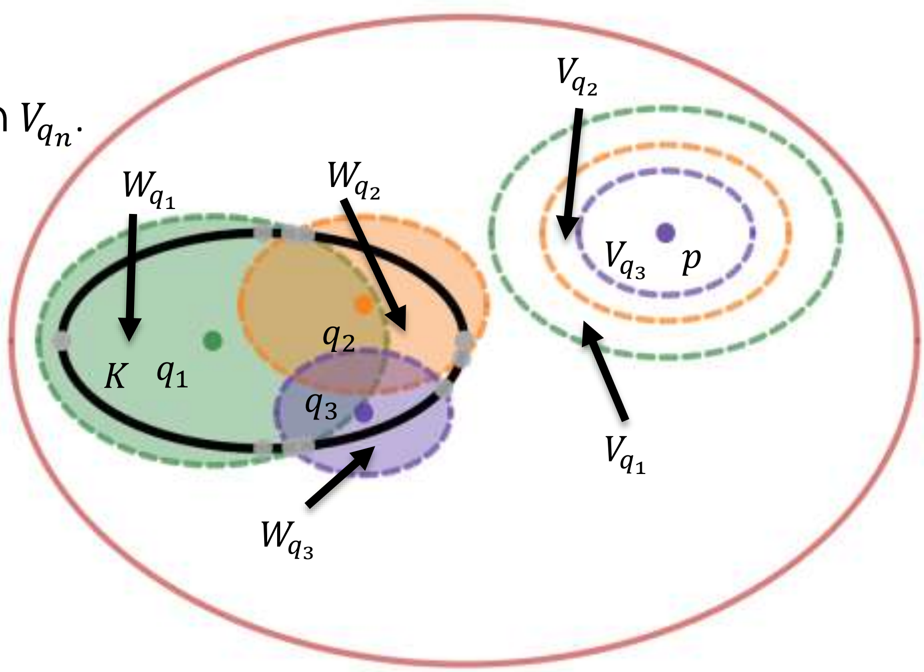
Thus we know $V_q \cap W_q = \emptyset$.

We also have that $K \subseteq \bigcup_{q \epsilon K} W_q$;

Since K is compact we know that there is some finite number of $q_i{}'s$ with:

$K \subseteq \bigcup_{i=1}^{n} W_{q_i}$.

Let $V = V_{q_1} \cap V_{q_2} \cap \ldots \cap V_{q_n}$.



V is an open set (finite intersection of open sets) containing p and V does not intersect K: If $x \epsilon K$ then $x \epsilon W_{q_i}$ for some i. But $V_{q_i} \cap W_{q_i} = \emptyset$. So $x \notin V_{q_i}$ and hence $x \notin V_{q_1} \cap V_{q_2} \cap \ldots \cap V_{q_n} = V$.

Thus $V \subseteq K^c$ and K^c is open.

Since K^c is open, K is closed.

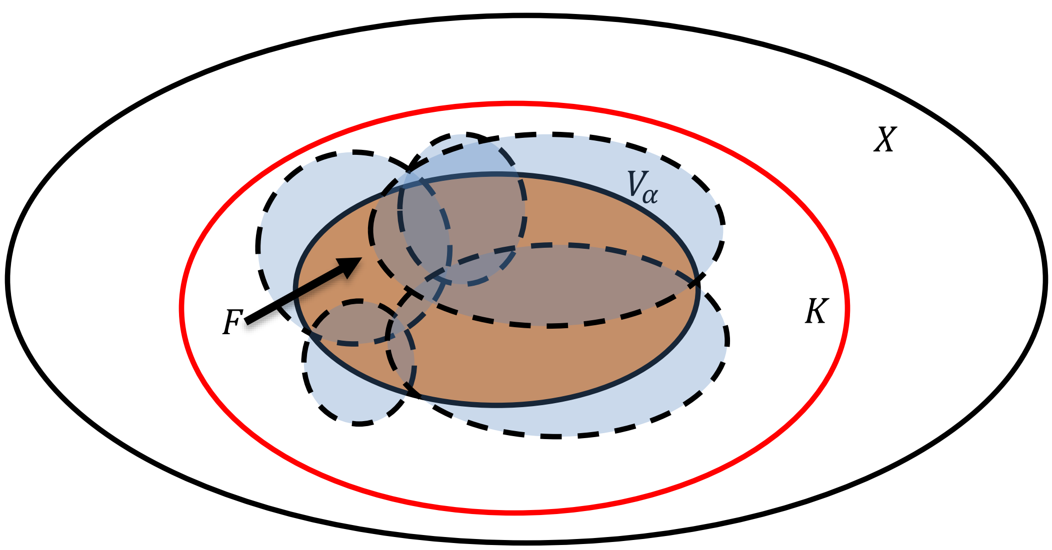
Theorem: Closed subsets of compact sets are compact.

Proof: Let F be a closed subset of a compact set $K \subseteq X, d$ a metric space.

Let $\{V_\alpha\}$ be an open cover of $F \subseteq K$.

Since F is closed, F^c is open.

$F^c \cup_\alpha V_\alpha$ is now an open cover of K (as well as F).



Since K is compact, we know there we only need a finite number of $F^c \cup_\alpha V_\alpha$ to cover K.

Case 1: $F \subseteq K \subseteq F^c \cup V_{\alpha_1} \cup V_{\alpha_2} \cup \ldots \cup V_{\alpha_n}$

But $F^c \cap F = \emptyset$, so F^c is not need to cover F.

Hence $F \subseteq V_{\alpha_1} \cup V_{\alpha_2} \cup \ldots \cup V_{\alpha_n}$, which is a finite subcover of $\{V_\alpha\}$

Case 2: $F \subseteq K \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \ldots \cup V_{\alpha_n}$ (we don't need F^c)

In this case $V_{\alpha_1} \cup V_{\alpha_2} \cup \ldots \cup V_{\alpha_n}$ is already a finite subcover of $\{V_\alpha\}$.

In either case we have shown that F is compact.

This theorem implies that finite intersections of compact sets are compact.

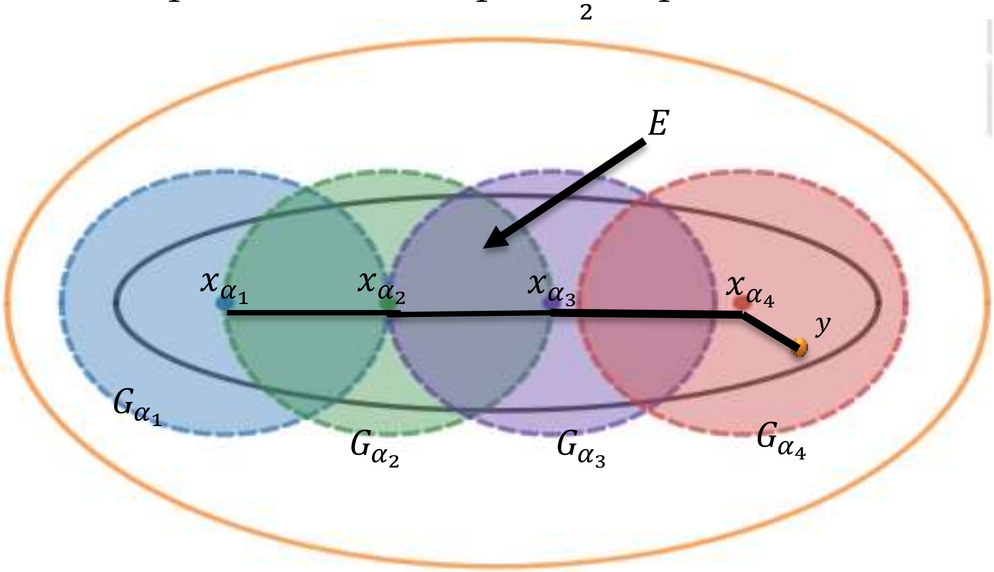
Theorem: Every compact subset E of a metric space X is bounded.

Proof: Let $\{G_\alpha\}$ be a collection of open sets $G_\alpha = N_{\frac{1}{2}}(x_\alpha)$, where $x_\alpha \epsilon E$.

Thus, $E \subseteq \bigcup_\alpha G_\alpha$.

Since E is compact we can find a finite subcover $E \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}$.

Take $x_{\alpha_1} \epsilon G_{\alpha_1}$, where x_{α_1} is the center of $G_{\alpha_1} = N_{\frac{1}{2}}(x_{\alpha_1})$.



Then if $y\epsilon E$, $y \in G_{\alpha_j}$, for some $1 \leq j \leq n$.

Then by the triangle inequality we have:

$$d\left(x_{\alpha_1}, y\right) \leq \sum_{i=1}^{j-1} d(x_{\alpha_i}, x_{\alpha_{i+1}}) + \frac{1}{2} \leq \sum_{i=1}^{n} d(x_{\alpha_i}, x_{\alpha_{i+1}}) + \frac{1}{2}$$

so E is bounded.

Heine-Borel Theorem: If E is a set in $\mathbb{R}^n$ then E is compact if and only if E is closed and bounded.

Ex. Identify which of the following subsets of $\mathbb{R}^2$ are compact:

$A = \{(x, y) |\ |x| \leq 2,\ \ |y| \leq 1\}$ Yes, closed and bounded

$B = \{(x, y) |\ |x| = 2, |y| \leq 1\}$ Yes, closed and bounded

$C = \{(x, y) |\ |x| = 2, |y| \geq 1\}$ No, not bounded

$D = \{(x, y) |\ 0 < x^2 + y^2 \leq 4\}$ No, not closed

$E = \{(x, y) ||x| = 3\}$ No, not bounded

$F = \{(0,0), (0,1), (1,0)\}$ Yes, closed and bounded

Connected Sets

Def. Two subsets A, B of a metric space X, d are said to be **separated** if $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$ (i.e., no point of A lies in the closure of B and no point of B lies in the closure of A).

Def. A set $E \subseteq X, d$ a metric space is said to be **connected** if E is not the union of two nonempty separated sets.

Ex. if $A = (0,1)$ and $B = (1,2)$, then A and B are separated sets since
$\bar{A} = [0,1], \bar{B} = [1,2]$
thus:
$$A \cap \bar{B} = (0,1) \cap [1,2] = \emptyset \text{ and}$$
$$\bar{A} \cap B = [0,1] \cap (1,2) = \emptyset.$$
Thus the set $A \cup B = (0,1) \cup (1,2)$ is not a connected set.

Ex. If $A = (0,1]$ and $B = (1,2)$, then A and B are not separated since $\bar{B} = [1,2]$ and thus

$$A \cap \bar{B} = (0,1] \cap [1,2] = \{1\} \neq \emptyset$$

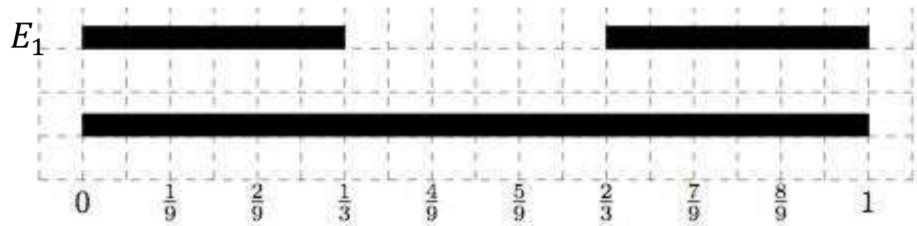
(notice that $\bar{A} \cap B = [0,1] \cap (1,2) = \emptyset$).

Theorem: A subset $E \subseteq \mathbb{R}$ is connected if and only if , if $x \epsilon E$, $y \epsilon E$ and $x < z < y$ then $z \epsilon E$.

Ex. The Cantor Set is a subset of the interval $[0,1]$, with very interesting properites. Here's how it's created.

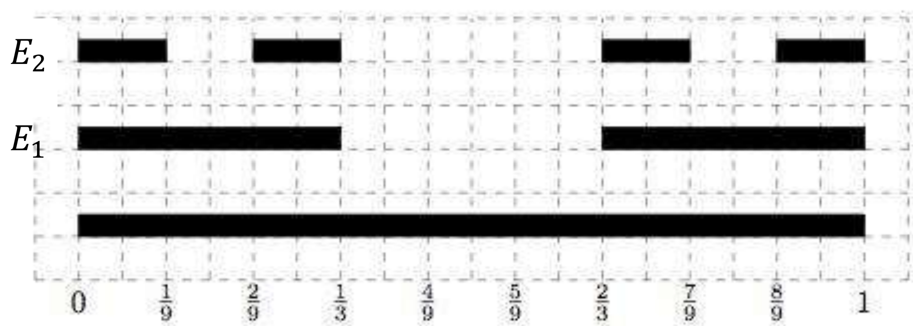
Let $I = [0,1]$. Remove the open middle third segment $(\frac{1}{3}, \frac{2}{3})$ and let

$E_1 = \left[0, \frac{1}{3}\right] \cup [\frac{2}{3}, 1]$.



Now remove the open middle thirds of each part above. Let

$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.



Continue this way always removing open middle thirds of each segment to get

$E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$. The Cantor set is defined to be:

$$C = \bigcap_{i=1}^{\infty} E_i$$

where E_n is the union of 2^n intervals, each of length 3^{-n}.

Properites of the Cantor set.

1. C is compact because since each of the E_i's is closed (a finite union of closed sets), $C = \bigcap_{i=1}^{\infty} E_i$ is also closed. But C is also a bounded subset of $\mathbb{R}$, so C is also compact.

2. Let $x = 0.a_1a_2a_3 \dots$ be a base 3 expansion of any number $x \in [0,1]$, i.e. $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$; where $a_i = 0, 1, or\ 2$. Then $x \in C$ if and only if $a_i = 0\ or\ 2$.

3. C is uncountable. Suppose C is countable, then $C = \{x_1, x_2, x_3, \dots\}$.
   $x_1 = 0.a_{11}a_{12}a_{13} \dots$
   $x_2 = 0.a_{21}a_{22}a_{23} \dots$
   $\vdots$
   $x_k = 0.a_{k1}a_{k2}a_{k3} \dots$
   $\vdots$
   where $a_{ij} = 0\ or\ 2$ for all i, j.

   Let $y = 0.b_1b_2b_3 \dots$
   where $b_i = 0$ if $a_{ii} = 2$
   $= 2$ if $a_{ii} = 0$
   Then y is not equal to x_k for any k, which is a contradiction.

4. C has "measure" (i.e., "length") 0.

   $[0,1]$ has length 1. Let's add up the lengths of the sets removed from $[0,1]$ to create the Cantor set.

   $$\text{Length of sets removed from } [0,1] = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots + \frac{2^{n-1}}{3^n} + \cdots$$
   $$= \frac{1}{3}\left(1 + \frac{2}{3} + \frac{4}{9} + \cdots + \frac{2^n}{3^n} + \cdots\right)$$
   $$= \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1.$$

   Thus, the Cantor set must have measure (i.e., length) 0. Thus the Cantor set is an uncountable set (i.e. it can be put in 1-1 correspondence with $[0,1]$) with measure 0!

5. C contains no intervals. That is, no subset of C is connected.

   If C did contain any interval $(a, b), b > a$, then the measure of C couldn't be 0 since the measure of (a, b) is $|b - a| > 0$.