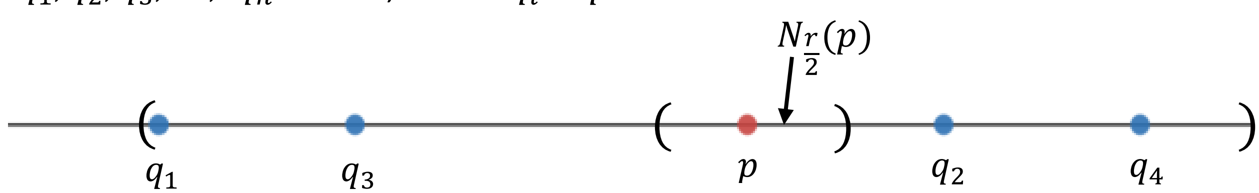
Open and Closed Sets in a Metric Space

Theorem: If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Proof: This is a proof by contradiction. We start by assuming that the theorem is false and derive a contradiction.

Assume that there is a neighborhood N of p, a limit point of E, that contains only a finite number of points of E. Let's call those points

$q_1, q_2, q_3, \dots, q_n \epsilon N \cap E$, where $q_i \neq p$ for all i.



Let $r = \min_{1 \leq m \leq n} \mathrm{d}(p, q_m) > 0$.

Then the neighborhood $N_{\frac{r}{2}}(p)$ contains no point in E.

But then, p is not a limit point of E (recall that p is a limit point of E if every neighborhood of p contains a point $q \neq p$ such that $q \epsilon E$).

This contradicts the assumption that p is a limit point of E.

Thus the assumption that there is a neighborhood N of p that contains only a finite number of points of E, is false.

Hence every neighborhood of p contains infinitely many points of E.

Theorem: A set E is open if and only if its complement is closed.

Proof: We have to prove 2 statements here:

1. If E^c is closed then E is open
2. If E is open then E^c is closed

Proof of #1: We assume E^c is closed and we have to prove that E is open.

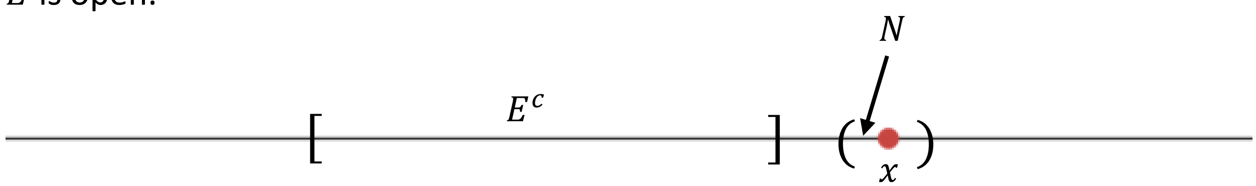
To prove E is open we need to show that given any point $x \epsilon E$, we can find a neighborhood of x, N, such that $N \subseteq E$.

Since $x \epsilon E$, $x \notin E^c$ (by definition of a complement)

Since E^c is closed (by assumption), E^c contains all of its limit points, thus x is not a limit point of E^c.

Since x is not a limit point of E^c, there exists a neighborhood, N, of x such that:

$N \cap E^c = \emptyset$. But this means $N \subseteq E$, and thus x is an interior point of E and E is open.



Proof of #2: We assume E is open and we need to prove E^c is closed.

To prove E^c is closed we will show that E^c contains all of its limit points (definition of a closed set).

Let x be a limit point of E^c. Then every neighborhood of x contains a point $y \epsilon E^c$.

Since every neighborhood of x contains a point $y\epsilon E^c$, x is not an interior point of E (if x were an interior point of E, you could find a neighborhood, N, of x such that $N \subseteq E$).

Since E is open, by assumption, $x \notin E$ (since all points of an open set are interior points).

Since $x \notin E$, by definition, $x\epsilon E^c$. Hence E^c is closed.

Corollary: A set F is closed if and only if F^c is open.

Proof: By the theorem we just proved, a set E is open if and only if E^c is closed.

This means that a set F^c is open if and only if $(F^c)^c = F$ is closed.

That's the same as saying: a set F is closed if and only if F^c is open.

This corollary is very useful. We will often prove a set is closed by showing that its complement is open.

Ex. Let $X = \mathbb{R}$ with the standard metric be a metric space

a. Prove $(0,1) \subseteq \mathbb{R}$ is an open set.

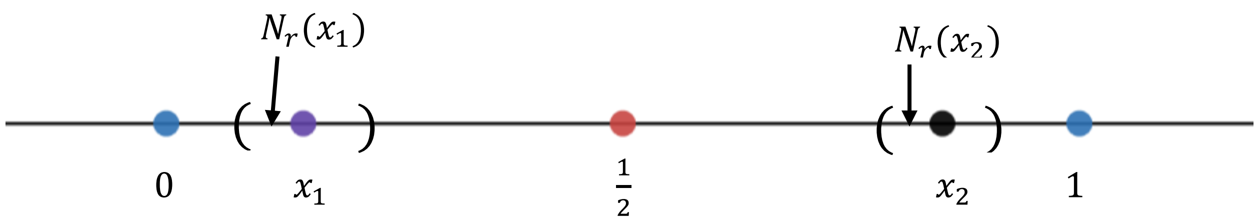
b. Prove $(-\infty, 0] \cup [1, \infty) \subseteq \mathbb{R}$ is a closed set

c. Prove $[0,1] \subseteq \mathbb{R}$ is a closed set.

a. To show $(0,1) \subseteq \mathbb{R}$ is an open set, we must show that for every $x \epsilon (0,1)$ we can find a neighborhood $N_r(x) \subseteq (0,1)$.

Since $0 < x < 1$, let $N_r(x) = N_{\frac{x}{2}}(x)$ if $0 < x \leq \frac{1}{2}$

$= N_{\frac{1-x}{2}}(x)$ if $\frac{1}{2} < x < 1$



Now we must show that $N_r(x) \subseteq (0,1)$.

If $0 < x \leq \frac{1}{2}$ then $N_{\frac{x}{2}}(x) = \{p \mid |x - p| < \frac{x}{2}\}$. That's the same as:

$-\frac{x}{2} < x - p < \frac{x}{2}$; Now subtract x from all quantities:

$-\frac{3x}{2} < -p < -\frac{x}{2}$ Multiply the inequality by -1

$\frac{3x}{2} > p > \frac{x}{2}$ Since $0 < x \leq \frac{1}{2}$ we have:

$\frac{3}{4} \geq \frac{3x}{2} > p > \frac{x}{2} > 0.$

So $N_{\frac{x}{2}}(x) \subseteq (0,1)$

If $\frac{1}{2} < x < 1$ then $N_{\frac{1-x}{2}}(x) = \{p| \quad |x - p| < \frac{1-x}{2}\}$. That's the same as:

$$-\left(\frac{1-x}{2}\right) < x - p < \frac{1-x}{2}$$ Which is the same as:

$$\frac{x-1}{2} < x - p < \frac{1-x}{2}$$ subtract x from all quantities:

$$\frac{-x-1}{2} < -p < \frac{1-3x}{2}$$ Multiply the inequality by -1

$$\frac{x+1}{2} > p > \frac{3x-1}{2}$$ Since $\frac{1}{2} < x < 1$ we have:

$$1 > \frac{x+1}{2} > p > \frac{3x-1}{2} > \frac{1}{4}.$$

So $N_{\frac{1-x}{2}}(x) \subseteq (0,1)$.

Thus $(0,1)$ is open.

b. To show $(-\infty, 0] \cup [1, \infty) \subseteq \mathbb{R}$ is a closed set, we just need to show that its complement is open. The complement of this set is $(0,1)$, which we just showed is open. Thus $(-\infty, 0] \cup [1, \infty) \subseteq \mathbb{R}$ is a closed set. We could also have shown that $(-\infty, 0] \cup [1, \infty) \subseteq \mathbb{R}$ contains all of its limit points.

c. To prove $[0,1] \subseteq \mathbb{R}$ is a closed set, we can either show that $[0,1]$ contains all of its limit points or show that its complement is open. This time we will show that the complement of $[0,1]$, $[0,1]^c$, is open.

To show that $[0,1]^c$ is open we must show that given any point $x \in [0,1]^c$ we can find a neighborhood, $N(x) \subseteq [0,1]^c$.

Let's assume that x is in the complement of $[0,1]$, so $x \notin [0,1]$.

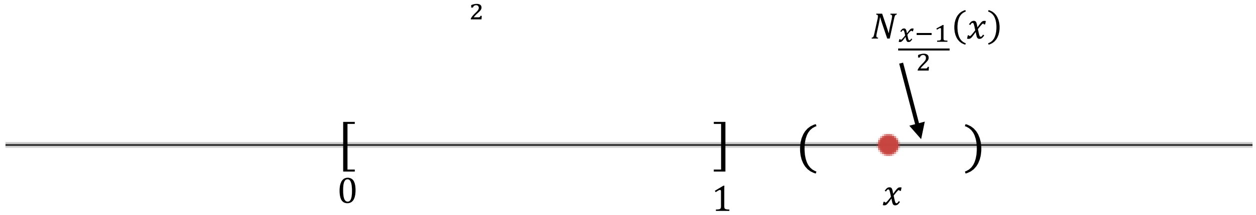
Since $x \notin [0,1]$, then either $x > 1$ or $x < 0$.

Case 1: $x > 1$

Take a neighborhood of x given by $N_{\frac{x-1}{2}}(x) = \{p \,||x - p| < \frac{x-1}{2}\}$

Note: $\frac{x-1}{2}$ is half the distance from x to 1.

Now we have to show that $N_{\frac{x-1}{2}}(x) \subseteq [0,1]^c$.



$N_{\frac{x-1}{2}}(x) = \{p \,||x - p| < \frac{x-1}{2}\}$ is the same as:

$-\frac{x-1}{2} < x - p < \frac{x-1}{2}$ which is the same as:

$\frac{1-x}{2} < x - p < \frac{x-1}{2}$ Subtract x from all quantities

$\frac{1-3x}{2} < -p < \frac{-x-1}{2}$ multiply by -1

$\frac{3x-1}{2} > p > \frac{x+1}{2}$ Since $x > 1$ we have:

$\frac{3x-1}{2} > p > \frac{x+1}{2} > 1$ So $p \notin [0,1]$

which means that $N_{\frac{x-1}{2}}(x) \subseteq [0,1]^c$.

Case 2: $x < 0$

Take a neighborhood of x given by $N_{\frac{|x|}{2}}(x) = \{p \mid |x - p| < \frac{|x|}{2}\}$

Note: $\frac{|x|}{2}$ is $\frac{1}{2}$ the distance from x to 0.

$N_{\frac{|x|}{2}}(x) = \{p \mid |x - p| < \frac{|x|}{2}\}$ is the same as:

$|x - p| < \frac{|x|}{2}$ since $x < 0$ we know $\frac{|x|}{2} = -\frac{x}{2}$

$|x - p| < \frac{-x}{2}$ which is the same as:

$\frac{x}{2} < x - p < \frac{-x}{2}$ Subtract x from all quantities

$\frac{-x}{2} < -p < \frac{-3x}{2}$ Multiply by -1

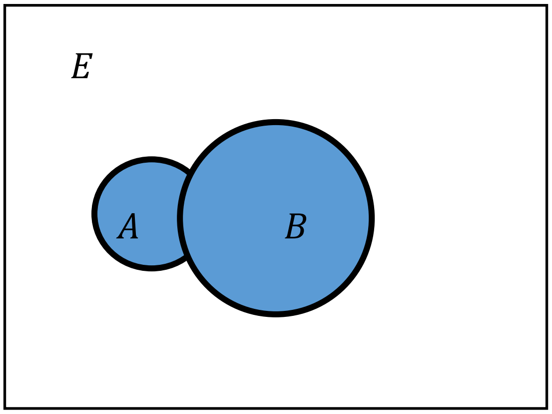
$\frac{x}{2} > p > \frac{3x}{2}$ since $x < 0$ we have

$0 > \frac{x}{2} > p > \frac{3x}{2}$ So $p \notin [0,1]$

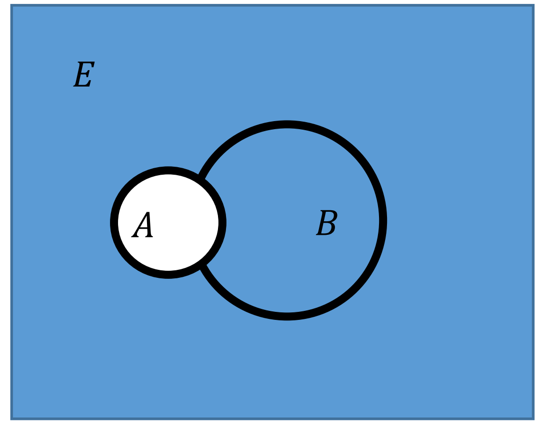
which means that $N_{\frac{|x|}{2}}(x) \subseteq [0,1]^c$

Thus $[0,1]^c$ is open.

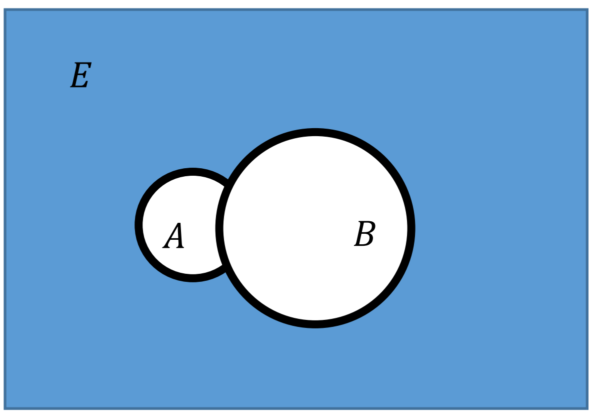
The next theorem is a generalization of one of De Morgan's laws, $(A \cup B)^c = A^c \cap B^c$, which is illustrated below.



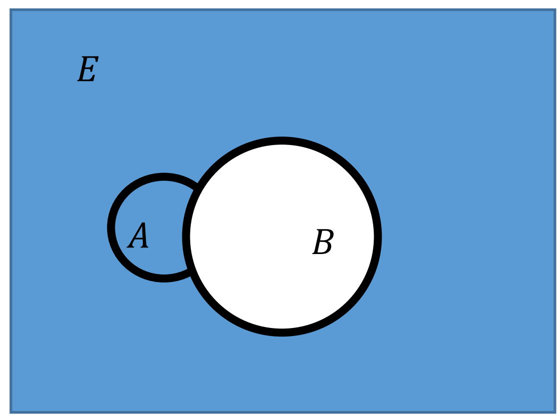
$A \cup B$



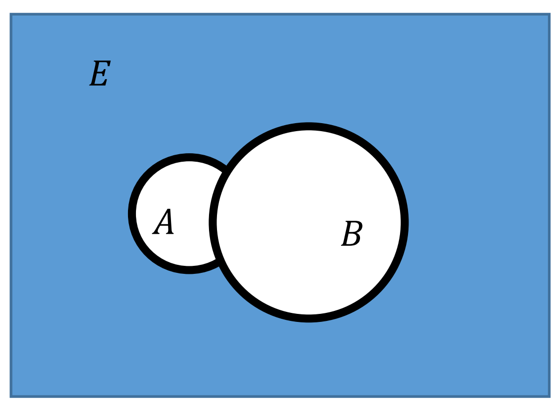
A^c



$(A \cup B)^c$



B^c



$A^c \cap B^c$

Therefore, $(A \cup B)^c = A^c \cap B^c$.

Theorem: Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α. Then

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c).$$

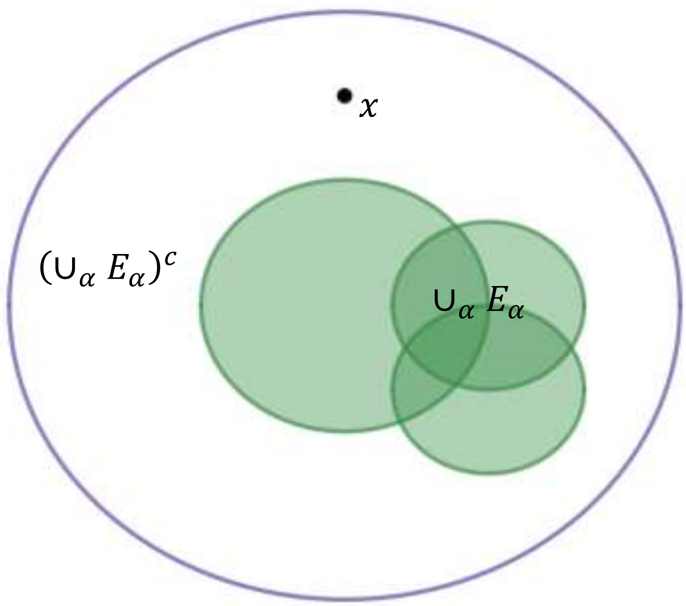
Proof: We will do this by first showing that $\left(\bigcup_\alpha E_\alpha\right)^c \subseteq \bigcap_\alpha (E_\alpha^c)$.

Then we will show that $\bigcap_\alpha (E_\alpha^c) \subseteq \left(\bigcup_\alpha E_\alpha\right)^c$.

This implies that $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c)$.

Let's show $\left(\bigcup_\alpha E_\alpha\right)^c \subseteq \bigcap_\alpha (E_\alpha^c)$.

If $x \epsilon \left(\bigcup_\alpha E_\alpha\right)^c$ then $x \notin \bigcup_\alpha E_\alpha$ by definition of a complement.



Thus $x \notin E_\alpha$ for any α (otherwise x would be in $\bigcup_\alpha E_\alpha$)

Therefore, $x \epsilon {E_\alpha}^c$ for all α.

That means that $x \epsilon \bigcap_\alpha {E_\alpha}^c$ (definition of intersection)

Hence $\left(\bigcup_\alpha E_\alpha\right)^c \subseteq \bigcap_\alpha (E_\alpha^c)$

Now let's show $\bigcap_\alpha (E_\alpha^c) \subseteq \left(\bigcup_\alpha E_\alpha\right)^c$.

Let $x \epsilon \bigcap_\alpha (E_\alpha^c)$. Then by definition of intersection, $x \epsilon E_\alpha^c$ for every α.

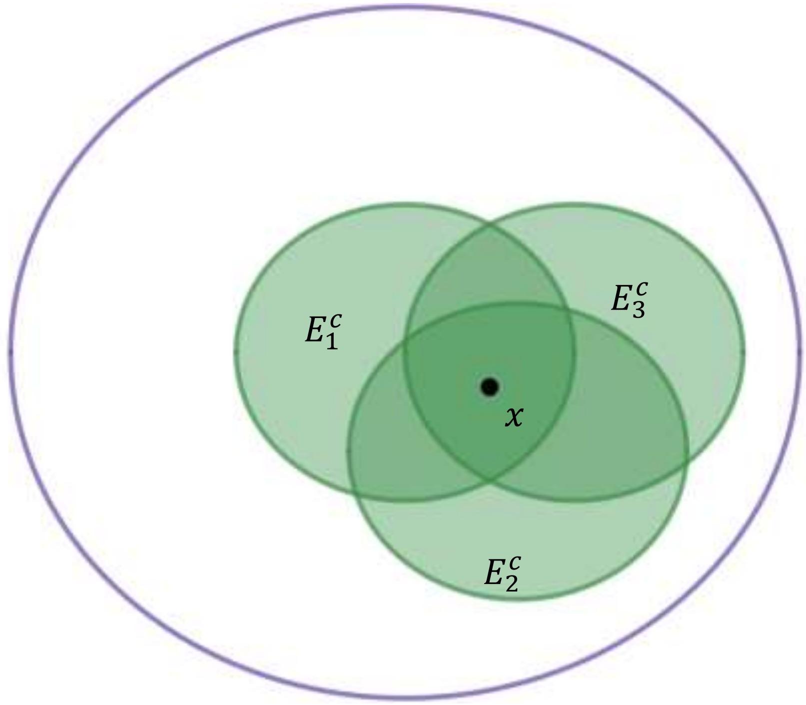
Thus $x \notin E_\alpha$ for any α.

That means that $x \notin \bigcup_\alpha E_\alpha$.

That means that $x \epsilon \left(\bigcup_\alpha E_\alpha\right)^c$ by the definition of complement.

Thus $\bigcap_\alpha (E_\alpha^c) \subseteq \left(\bigcup_\alpha E_\alpha\right)^c$

Hence, $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c)$.



Theorem:

a. For any collection $\{G_\alpha\}$ of open sets (this could be a finite collection or infinite, even uncountably infinite collection) $\bigcup_\alpha G_\alpha$ is open.

b. For any collection $\{F_\alpha\}$ of closed sets $\bigcap_\alpha F_\alpha$ is closed.

c. For any finite collection of open sets $\{G_i\}$, $\bigcap_{i=1}^n G_i$ is open.

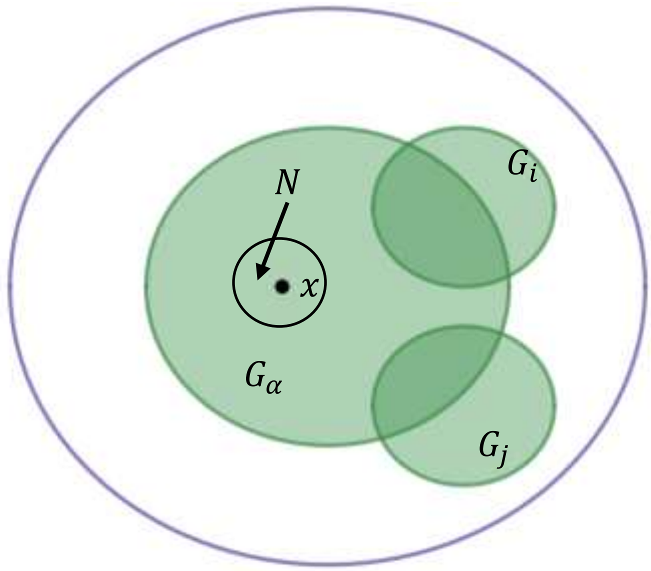
d. For any finite collection of closed sets $\{F_i\}$, $\bigcup_{i=1}^n F_i$ is closed.

Proof:

a. Let $G = \bigcup_\alpha G_\alpha$, where G_α is open for all α. To show G is open we need to show that for any $x \epsilon G$ we can find a neighborhood of x that lies completely in G.

If $x \epsilon G$, then x must lie in some open set G_α.

Since G_α is open, there exists some neighborhood N of x that lies entirely in G_α.



That neighborhood N also lies in $G = \bigcup_\alpha G_\alpha$.

Hence G is open.

b. To prove for any collection $\{F_\alpha\}$ of closed sets $\bigcap_\alpha F_\alpha$ is closed, we note that the previous theorem stated that for a collection of sets $\{E_\alpha\}$,

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c).$$

Now let $E_\alpha = {F_\alpha}^c$ and hence $E_\alpha^c = F_\alpha$.

Now substituting into the above equation:

$$\left(\bigcup_\alpha {F_\alpha}^c\right)^c = \bigcap_\alpha (F_\alpha).$$

Since F_α is closed for all α, ${F_\alpha}^c$ must be open for all α.

By part "a" of this theorem, $\bigcup_\alpha {F_\alpha}^c$ is also open.

Since $\bigcup_\alpha {F_\alpha}^c$ is open, $\left(\bigcup_\alpha {F_\alpha}^c\right)^c$ must be closed since it's the complement of an open set.

Thus, since $\left(\bigcup_\alpha {F_\alpha}^c\right)^c = \bigcap_\alpha (F_\alpha)$, $\bigcap_\alpha (F_\alpha)$ is also closed.

c. To show that for any finite collection of open sets $\{G_i\}$, $\bigcap_{i=1}^n G_i$ is open, it helps to think about the argument for 2 open sets (it is often helpful when trying to prove most propositions involving n "things" to see how the argument works when $n = 2$).

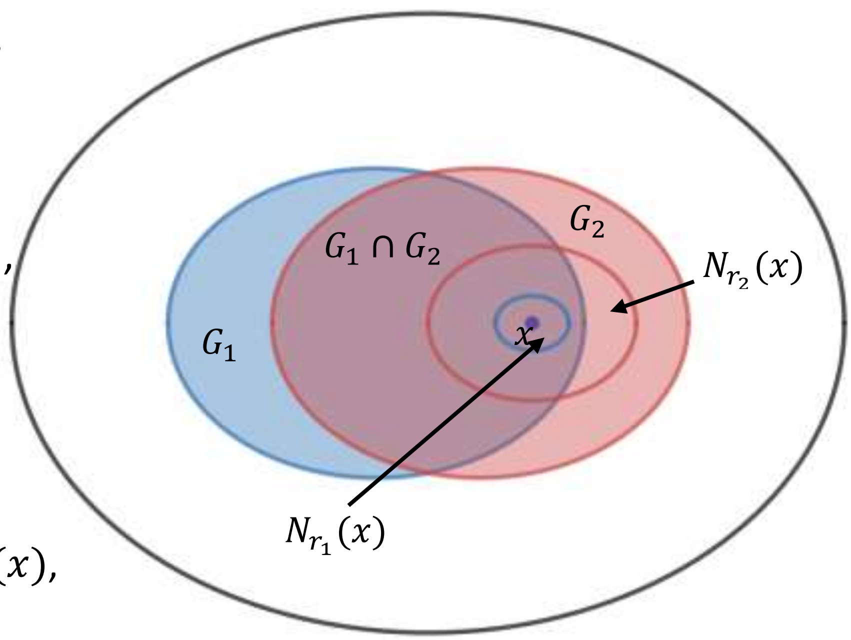
Suppose G_1 and G_2 are open sets. Let's show $G_1 \cap G_2$ is open.

If $G_1 \cap G_2$ (or $\bigcap_{i=1}^n G_i$) is empty, then $G_1 \cap G_2$ (or $\bigcap_{i=1}^n G_i$) is open.

If $G_1 \cap G_2$ is not empty, then let's choose any point $x \epsilon G_1 \cap G_2$ and show it's an interior point (and hence $G_1 \cap G_2$ is open).

Since $x \epsilon G_1 \cap G_2$, $x \epsilon G_1$ and $x \epsilon G_2$.

Because G_1 is open, we can find neighborhood around $x \epsilon G_1$, $N_{r_1}(x)$, such that $N_{r_1}(x) \subseteq G_1$.



Because G_2 is open, we can find a neighborhood around $x \epsilon G_2$, $N_{r_2}(x)$, such that $N_{r_2}(x) \subseteq G_2$.

Now let $r = \min(r_1, r_2)$. We now have that $N_r(x) \subseteq G_1$, and $N_r(x) \subseteq G_2$.

So $N_r(x) \subseteq G_1 \cap G_2$. This means that x is an interior point for $G_1 \cap G_2$, and hence $G_1 \cap G_2$ is open.

To prove that for any finite collection of open sets $\{G_i\}$, $\bigcap_{i=1}^{n} G_i$ is open, we use the same argument but now $r = \min(r_1, r_2, \dots, r_n)$.

d. To show that for any finite collection of closed sets $\{F_i\}$, $\bigcup_{i=1}^{n} F_i$ is closed, we can use the fact that a set is closed if and only if its complement is open. So let's show that $(\bigcup_{i=1}^{n} F_i)^c$ is open.

From an earlier theorem we know that $(\bigcup_{i=1}^{n} F_i)^c = \bigcap_{i=1}^{n}({F_i}^c)$.

Since each F_i is close, we know that each ${F_i}^c$ must be open.

From part c we know that the finite intersection of open sets is open, thus $\bigcap_{i=1}^{n}({F_i}^c)$ is open.

Since $(\bigcup_{i=1}^{n} F_i)^c = \bigcap_{i=1}^{n}({F_i}^c)$, $(\bigcup_{i=1}^{n} F_i)^c$ is open.

Now we know that means that $\bigcup_{i=1}^{n} F_i$ is closed.

Ex. We need finiteness in both c and d in the previous theorem.

$\bigcap_{i=1}^{\infty}(-\frac{1}{i}, \frac{1}{i}) = \{0\}$ which is not open.

$\bigcup_{i=1}^{\infty}[\frac{1}{i}, 1] = (0,1]$ which is not closed.

Def. Let X, d be a metric space. If $E \subseteq X$, and if E' denotes the set of all limit points of E in X, then the **closure** of E, $\bar{E}$=$E \cup E'$.

Theorem: If X, d is a metric space and $E \subseteq X$, then

a. $\bar{E}$ is closed

b. $E = \bar{E}$ if and only if E is closed

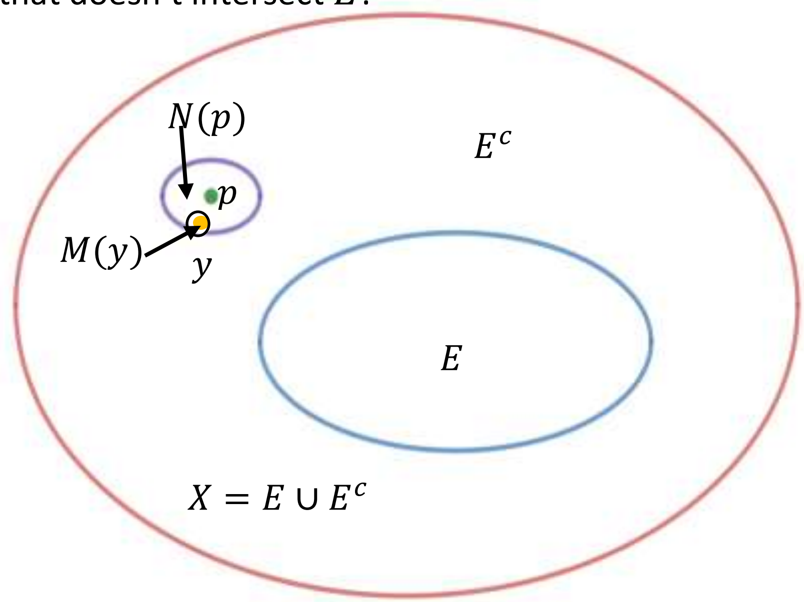
c. $\bar{E} \subseteq F$ for every closed set $F \subseteq X$ such that $E \subseteq F$.

Proof:

a. To show $\bar{E}$ is closed, we will show that $(\bar{E})^c$ is open.

Let's choose any point $p \epsilon X$ where $p \notin \bar{E}$ (ie $p \epsilon (\bar{E})^c$). We just need to show that p is an interior point.

Since $\bar{E}$=$E \cup E'$, p is neither in E nor a limit point of E. Hence there is a neighborhood of p, $N(p)$, that doesn't intersect E.



$N(p)$ can't intersect E' either. To see this, suppose there is a $y \in E' \cap N(p)$, then there is a neighborhood around y, $M(y)$, that lies inside of $N(p) \subseteq E^c$, since $N(p)$ is open.

But since y is a limit point of E, $M(y)$ must intersect E, which means that $N(p) \subseteq E^c$ intersects E, which is a contradiction ($E \cap E^c = \emptyset$, by definition).

Thus that neighborhood $N(p)$ lies completely in $(\bar{E})^c$, hence $(\bar{E})^c$ is open and thus $\bar{E}$ is closed.

b. First we show if $E = \bar{E}$ then E is closed. That's follows from part a.

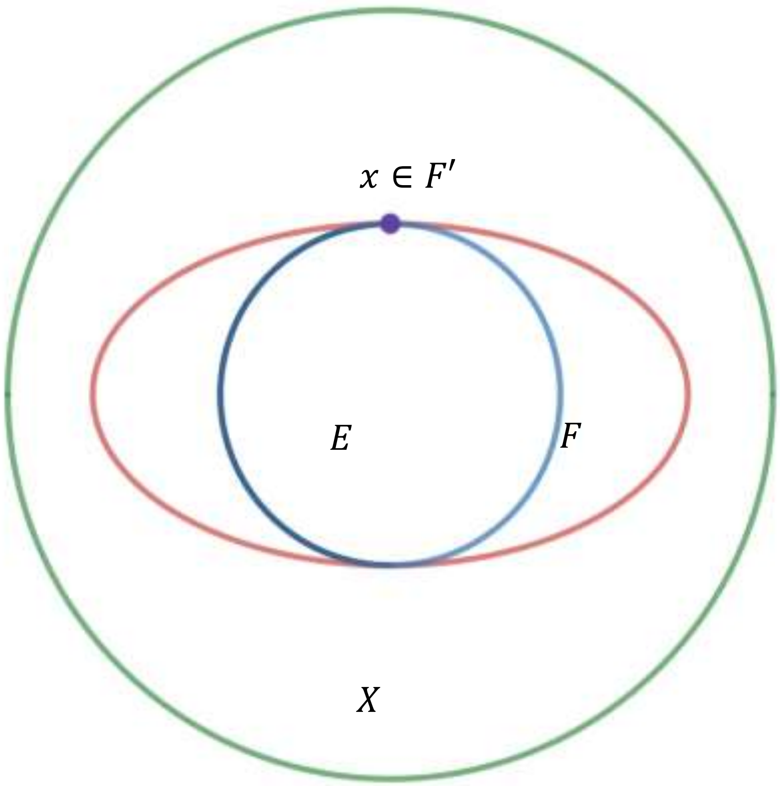
Now we show if E is closed then $E = \bar{E}$.

If E is closed then E contains all of its limit points (by definition).

Hence $E = E \cup E'$ =$\bar{E}$.

c. To show that $\bar{E} \subseteq F$ for every closed set $F \subseteq X$ such that $E \subseteq F$, assume that F is a closed set such that $E \subseteq F \subseteq X$.

Since F is closed, it contains all of its limit points, F'.



But since $E \subseteq F$, any limit point of E is also a limit point of F.

Hence F contains all limit point of E, thus $\bar{E} \subseteq F$.