Metric Spaces: Definitions and Examples

Def. A set X, whose elements are called points, is said to be a **Metric Space** if for any two points $p, q \epsilon X$ there is a real number $d(p,q)$, called the distance, such that:

a. $d(p,q) > 0$ if $p \neq q$, and $d(p,p) = 0$.

b. $d(p,q) = d(q,p)$

c. $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \epsilon X$ (Triangle inequality)

any function satisfying a,b,c is called a **distance function** or **metric** on X.

Ex. Let $X = \mathbb{R}$, $d(p,q) = |p-q|$ (the standard distance function on $\mathbb{R}$). Show X, d is a metric space.

To show X, d is a metric space, we need to show that d satisfies a,b,c above.

a. $d(p,q) = |p-q| > 0$ if $p \neq q$ (property of absolute values),

$d(p,p) = |p-p| = 0$

b. $d(p,q) = |p-q| = |q-p| = d(q,p)$

c. We need to show $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \epsilon X$. For this distance function it means we need to show:

$$|p-q| \leq |p-r| + |r-q|$$

Proof: First we will show for all real numbers that $|x+y| \leq |x| + |y|$ (This is called the triangle inequality for real numbers. We will use it a lot.)

$$|x+y|^2 = (x+y)^2 = x^2 + 2xy + y^2 = |x|^2 + 2xy + |y|^2$$

But for any real number, w, $w \le |w|$; therefore:

$$|x+y|^2 = |x|^2 + 2xy + |y|^2 \le |x|^2 + 2|x||y| + |y|^2 = (|x|+|y|)^2$$

Taking square roots we get: $|x+y| \le |x| + |y|$.

Now we let $x = p - r$ and $y = r - q$. Notice that $x + y = p - q$.

$$|p-q| \le |p-r| + |r-q|$$

Thus $X = \mathbb{R}$, $d(p,q) = |p-q|$ is a metric space.

Ex. Let $X = \mathbb{R}^n$, $d(p,q) = \sqrt{(p_1-q_1)^2 + \cdots + (p_n-q_n)^2} = \|p-q\|$;

where $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$. Show X, d is a metric space.

($d(p,q) = \sqrt{(p_1-q_1)^2 + \cdots + (p_n-q_n)^2}$ is the standard metric on $\mathbb{R}^n$) .

We need to show that $d(p,q)$ satisfies a,b,c in the definition of a metric space.

a. $d(p,q) = \sqrt{(p_1-q_1)^2 + \cdots + (p_n-q_n)^2} > 0$; if $p \ne q$; since the expression under the square root sign is strictly positive if $p \ne q$.

$$d(p,p) = \sqrt{(p_1-p_1)^2 + \cdots + (p_n-p_n)^2} = 0$$

b. $d(p,q) = \sqrt{(p_1-q_1)^2 + \cdots + (p_n-q_n)^2}$

$$= \sqrt{(q_1-p_1)^2 + \cdots + (q_n-p_n)^2} = d(q,p)$$

c. We need to show $d(p,q) \le d(p,r) + d(r,q)$ for any $r \epsilon X$.

Proof: Once again we will start by showing $\|x+y\| \le \|x\| + \|y\|$, where $x, y \epsilon \mathbb{R}^n$ and $\|x\|$ mean taking the length of the vector $x = < x_1, \ldots, x_n >$.

$$\|x+y\|^2 = (x+y)\cdot(x+y) = x\cdot x + 2x\cdot y + y\cdot y$$
$$= \|x\|^2 + 2x\cdot y + \|y\|^2$$

Recall that: $x \cdot y = \|x\|\|y\|cos\theta \le \|x\|\|y\|$; since $|cos\theta| \le 1$.

$$\|x+y\|^2 = \|x\|^2 + 2x\cdot y + \|y\|^2 \le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$
$$= (\|x\| + \|y\|)^2$$

Taking square roots we get: $\|x+y\| \le \|x\| + \|y\|$.

Now let $x = p - r$ and $y = r - q$ (again: $x + y = p - q$) and substitute:

$\|p-q\| \le \|p-r\| + \|r-q\|$ or $d(p,q) \le d(p,r) + d(r,q)$ for any $r\epsilon X$.

Thus $X = \mathbb{R}^n$, $d(p,q) = \sqrt{(p_1-q_1)^2 + \cdots + (p_n-q_n)^2}$, is a metric space.

Notice that every subset $E \subseteq X, d$ of a metric space is again a metric space with the same distance function.

Ex. $X = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ is a metric space with $d(p,q) = |p-q|$

Ex. Let X be a non-empty set and d given by $d(p,q)=1$ if $p\neq q$, and 0 if $p=q$.

Prove that X,d is a metric space.

a. Notice $d(p,q)=1>0$ if $p\neq q$, and $d(p,p)=0$.

b. $d(p,q)=1=d(q,p)$ if $p\neq q$

c. By definition $d(p,q)\leq 1$. Unless $p=q=r$,

$$d(p,r)+d(r,q)\geq 1\geq d(p,q).$$

If $p=q=r$, then $d(p,q)=0$ and $d(p,r)+d(r,q)=0$, hence:

$d(p,q)\leq d(p,r)+d(r,q)$ for any $r\epsilon X$.

So X,d is a metric space.

Ex. Show $\mathbb{R},d$ is a metric space where $d(p,q)=|e^p-e^q|$.

a. $d(p,q)=|e^p-e^q|>0$ for $p\neq q$ and $d(p,p)=|e^p-e^p|=0$.

b. $d(p,q)=|e^p-e^q|=|e^q-e^p|=d(q,p)$.

c. We need to show: $d(p,q)\leq d(p,r)+d(r,q)$ for any $r\epsilon\mathbb{R}$. In this case:

$$|e^p-e^q|\leq|e^p-e^r|+|e^r-e^q|.$$

This looks daunting, but remember the Triangle inequality for real numbers:

$$|x+y|\leq|x|+|y|\ .$$

Now let $x = e^p - e^r$ and $y = e^r - e^q$, so $x + y = e^p - e^q$. Hence:

$$|e^p - e^q| \leq |e^p - e^r| + |e^r - e^q|.$$

Hence $\mathbb{R}, d$ is a metric space.

Ex. Let $d(p,q) = |e^p - e^q|$ be a metric on $\mathbb{R}$. Find the set of points $p \in \mathbb{R}$ such that $d(p,0) < \frac{1}{2}$.

$$d(p,0) = |e^p - e^0| = |e^p - 1| < \frac{1}{2}.$$

This last inequality is equivalent to:

$$-\frac{1}{2} < e^p - 1 < \frac{1}{2}$$ Now add 1 to all quantities:

$$\frac{1}{2} < e^p < \frac{3}{2}.$$ Now take natural logs of all quantities:

$$\ln\left(\frac{1}{2}\right) < p < \ln\left(\frac{3}{2}\right).$$

Thus the set of points $p \in \mathbb{R}$ such that $d(p,0) < \frac{1}{2}$ is: $\ln\left(\frac{1}{2}\right) < p < \ln\left(\frac{3}{2}\right)$.

Ex. Show $\mathbb{R}, d$ where $d(p,q) = |\sin(p-q)|$, is NOT a metric space.

a. $d(p,q) = |\sin(p-q)| \geq 0$, however, $d(0,\pi) = |\sin(0-\pi)| = 0$.
So there exist a $p \neq q$ where $d(p,q) = 0$, which violates $d(p,q) > 0,\ p \neq q$.

so $\mathbb{R}, d$ is not a metric space.

Note: Not all metric spaces are subsets of $\mathbb{R}^n$.

Ex. $X = C[0,1]$ =set of real valued, continuous functions on $[0,1]$. X is a metric space with either of these 2 metrics (there are an infinite number of metrics on X)

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|dx$$

$$d_2(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|.$$

Ex. Let $f(x) = x^2$ and $g(x) = x^3$. Notice that $f(x), g(x) \in C[0,1]$. Using the 2 metrics just defined on $C[0,1]$, find $d_1(f,g)$ and $d_2(f,g)$.

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|dx = \int_0^1 |x^2 - x^3|dx \,.$$

Notice that when $0 \le x \le 1,\ x^2 \ge x^3$ (when $0 < x < 1$ the higher the power the lower the value).

Thus when $0 \le x \le 1,\ x^2 - x^3 \ge 0$ hence $|x^2 - x^3| = x^2 - x^3$. So

$$d_1(f,g) = \int_0^1 |x^2 - x^3|dx = \int_0^1 (x^2 - x^3)dx$$

$$= \frac{1}{3}x^3 - \frac{1}{4}x^4\Big|_{x=0}^{x=1} = \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{12}\,.$$

$$d_2(f,g) = \max_{x\in[0,1]} |f(x) - g(x)| = \max_{x\in[0,1]} |x^2 - x^3|$$

To find the maximum value of $|h(x)|$, we need to find where $h(x)$ has its greatest positive value and its most negative value and choose the one which is greater in absolute value (e.g. if $h(x)$ has 4 as its most positive value and -6 as its most negative value then the maximum of $|h(x)|$ is |-6|=6.)

In this case we already saw that $x^2 - x^3 \geq 0$ so we just have to find the absolute maximum value of $h(x) = x^2 - x^3$. Using first year calculus, find the values of $h(x)$ at all critical points in [0,1] and then test the values at the endpoints.

$$h'(x) = 2x - 3x^2 = x(2 - 3x) = 0$$

$$\Rightarrow \qquad x = 0 \text{ or } x = \frac{2}{3}.$$

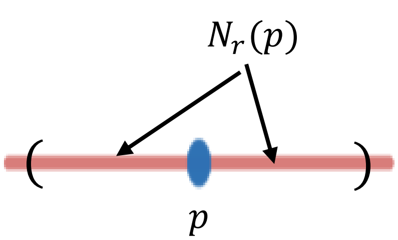
$$h(0) = 0\,, \quad h\left(\frac{2}{3}\right) = (\frac{2}{3})^2 - \left(\frac{2}{3}\right)^3 = \frac{4}{9} - \frac{8}{27} = \frac{4}{27}\,, \quad h(1) = 1^2 - 1^3 = 0.$$

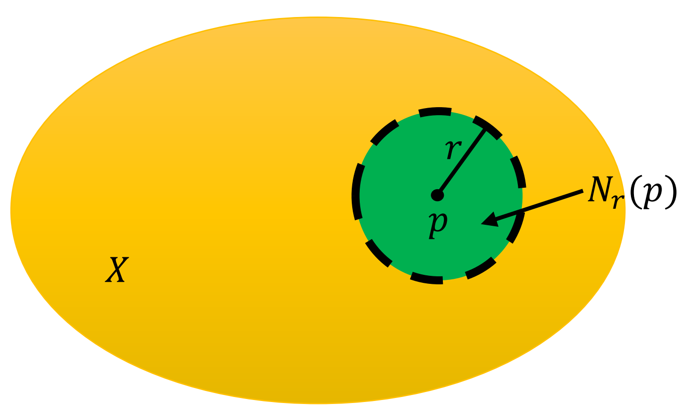
So the absolute maximum value of $h(x)$ is $\frac{4}{27}$ (absolute minimum is 0). So

$$d_2(f,g) = \max_{x\in[0,1]} |f(x) - g(x)| = \max_{x\in[0,1]} |x^2 - x^3| = \frac{4}{27}.$$

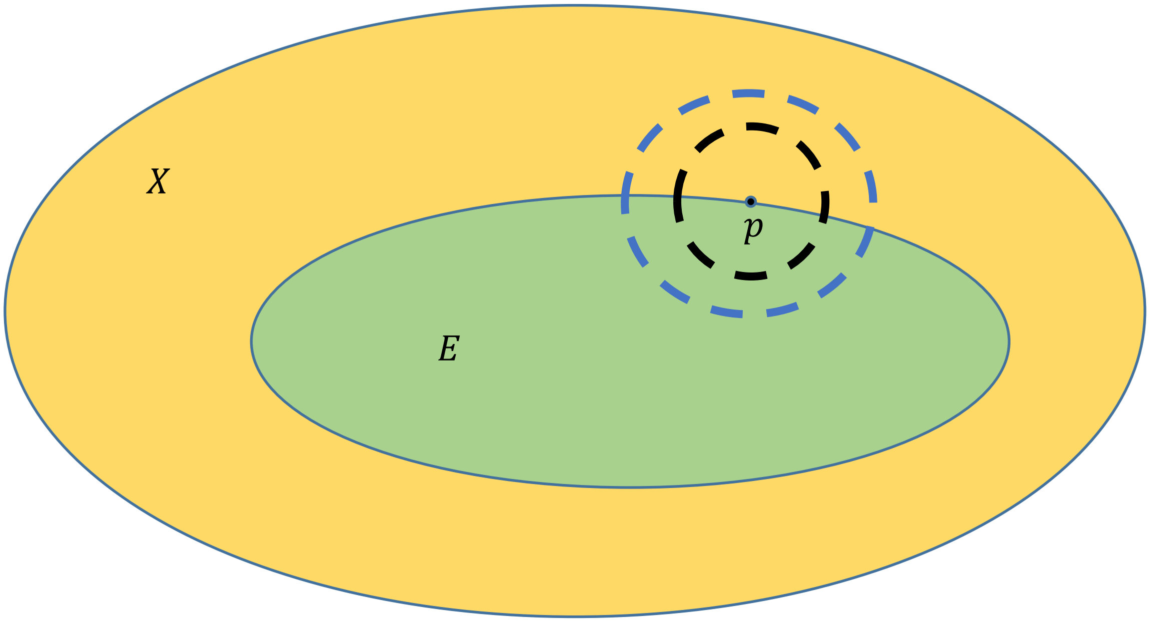
Def. Let X be a metric space with distance function d.

a. A **neighborhood of p**, where $p \epsilon X$, is a set $N_r(p)$ of all points q such that $d(p,q) < r$ for some $r > 0$.





b. A point p is a **limit point** of $E \subseteq X$ if every neighborhood of p contains a point $q \neq p$ such that $q \epsilon E$.

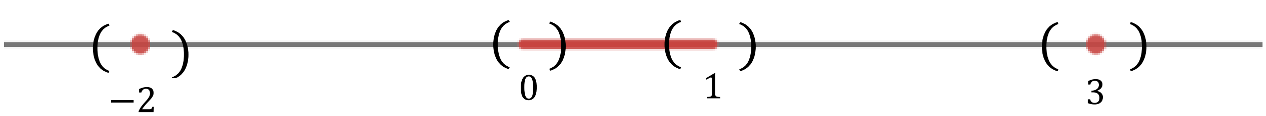


Ex. Let $X = \mathbb{R}$, and d the standard metric (i.e. $d(p,q) = |p - q|$). Let $E = (0,1)$

That is $E = \{x \epsilon \mathbb{R} |\ 0 < x < 1\}$. The set of limit points of $E = [0,1]$.

Ex. Let $X = \mathbb{R}$, and d the standard metric. Let $E = (0,1) \cup \{3\} \cup \{-2\}$.

The set of limit points of $E = [0,1]$.



c. If $p \epsilon E$ and p is not a limit point of E, then p is called an **isolated point** of E

Ex. Let $X = \mathbb{R}$, and d the standard metric. Let $E = (0,1) \cup \{3\} \cup \{-2\}$.

$\{3\}, \{-2\}$ are isolated points of E.

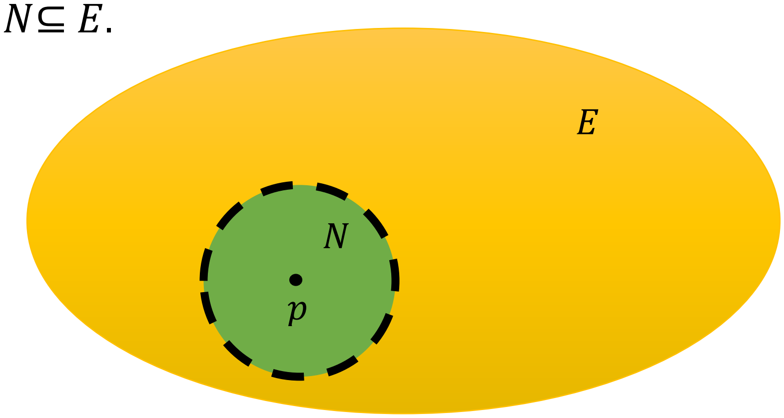
d. E is **closed** if every limit point of E is a point of E.

Ex. Let $X = \mathbb{R}$, and d the standard metric. Let $E = [0,1] \cup \{5\} \cup \{-3\}$.

E is closed in $X = \mathbb{R}$.

Let $F = (0,1] \cup \{5\} \cup \{-3\}$. F is not closed in $X = \mathbb{R}$, since $x = 0$ is a limit point of F, but is not contained in F.

e. A point p is an **interior point** of E if there is some neighborhood N of p, such that $N \subseteq E$.



Ex. Let $X = \mathbb{R}$, and d the standard metric. Let $E = [0,1) \cup \{3\}$.

$0 < x < 1$ are interior points of E. $x = 0$ and $x = 3$ are not interior points of E.

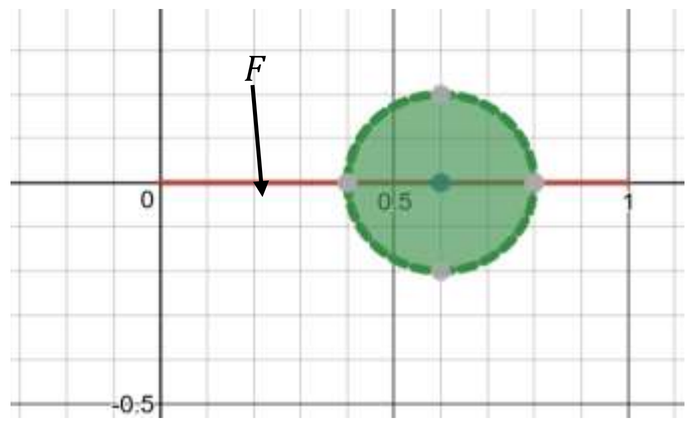
f. E is **open** if every point of E is an interior point

Ex. Let $X = \mathbb{R}$, and d the standard metric. Let $E = (0,1)$.

E is an open set in $X = \mathbb{R}$.

Note: Let $X = \mathbb{R}^2$, and d the standard metric. Let $F = \{(x, y)|\ 0 < x < 1, y = 0\}$

Although F is essentially the same set as E in our example, F is NOT an open subset of $X = \mathbb{R}^2$ because a neighborhood in $\mathbb{R}^2$ is a disk.

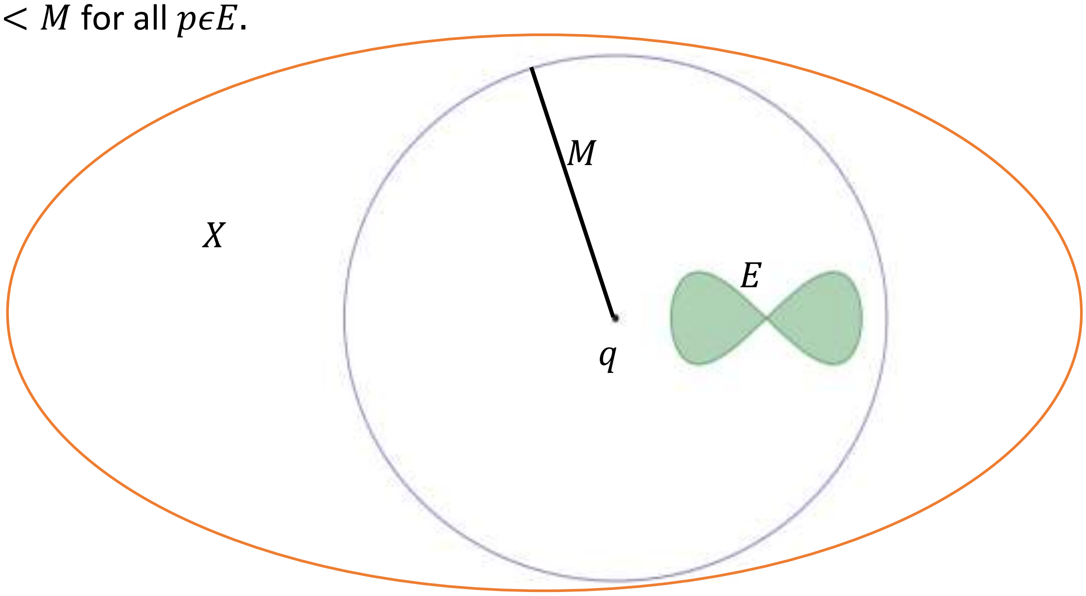


g. The **complement of E** (denoted E^c), is the set of all point $p \epsilon X$ such that $p \notin E$.

Ex. Let $X = \mathbb{R}$, and d the standard metric. Let $E = [0,1)$.

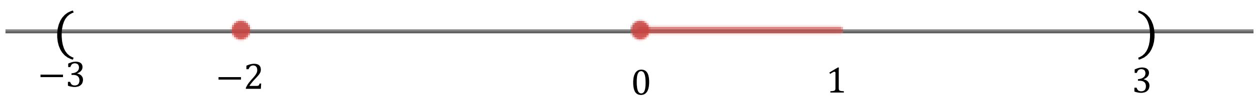
$E^c = (-\infty, 0) \cup [1, \infty)$.

h. E is **bounded** if there is a real number M and a point $q \epsilon X$ such that $d(p,q) < M$ for all $p \epsilon E$.



Ex. Let $X = \mathbb{R}$, and d the standard metric. Let $E = [0,1) \cup \{-2\}$.

E is a bounded set. We can take $0 \epsilon X$ and $d(0, p) < 3$, for all $p \epsilon E$.



i. E is **dense** in X if every point in X is a limit point of E, or a point of E (or both).

Ex. $E = \bigcup_{i=-\infty}^{i=\infty} (i, i+1)$, E is dense in $X = \mathbb{R}$, and d the standard metric.

Ex. $E = \{rational\ numbers\}$; E is dense in $X = \mathbb{R}$, and d the standard metric.

Ex. Consider the following subsets of the metric space $X = \mathbb{R}$, with d the standard metric.

$A = \{0,1,2,3, \ldots\}$, $B = \{0,1,2\}$, $C = \{x \mid |x| \leq 2\}$,

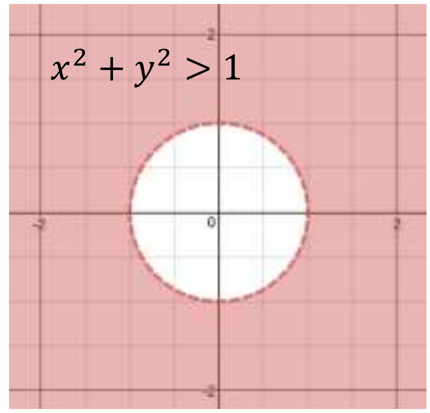
$D = \{x \mid -1 < x \leq 1\}$, $E = \{x \mid x \geq 0 \ \ or \ \ x = -2\}$, $F = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

Then we have the following table:

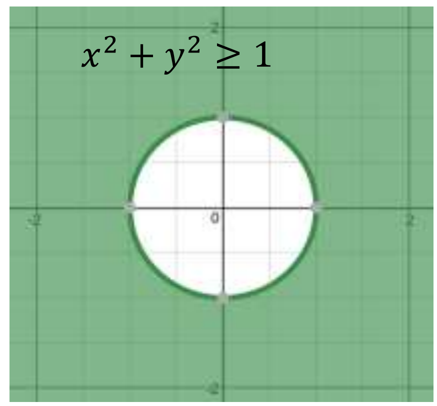
	Limit Points	Isolated Points	Bounded	Closed	Open
A	$\emptyset$	$\{0,1,2, \ldots\}$	N	Y	N
B	$\emptyset$	$\{0,1,2\}$	Y	Y	N
C	$[-2,2]$	$\emptyset$	Y	Y	N
D	$[-1,1]$	$\emptyset$	Y	N	N
E	$[0, \infty)$	$\{-2\}$	N	Y	N
F	$\{0\}$	$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$	Y	N	N

Ex. Consider the following subsets of the metric space $X = \mathbb{R}^2$ with d the standard metric.

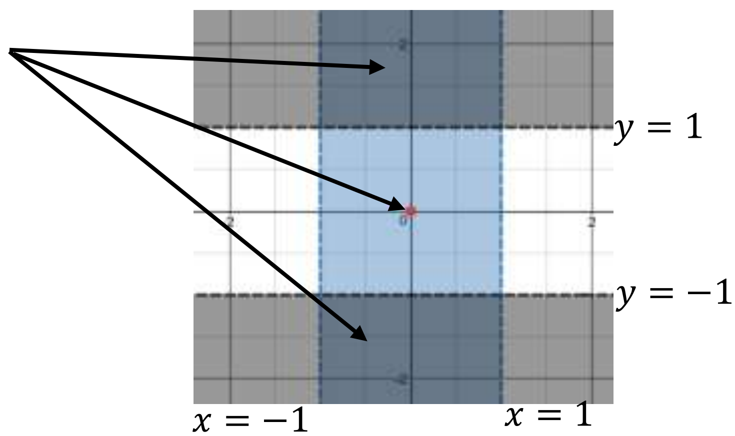
$A = \{(x, y) |\ x^2 + y^2 > 1\}$,



$B = \{(x, y) |\ x^2 + y^2 \geq 1\}$,



$C = \{(x, y) |\ |x| < 1,\ |y| > 1\} \cup \{(0,0)\}$



	Limit Points	Isolated Points	Bounded	Closed	Open
A	$x^2 + y^2 \geq 1$	$\emptyset$	N	N	Y
B	$x^2 + y^2 \geq 1$	$\emptyset$	N	Y	N
C	$\lvert x\rvert \leq 1$ and $\lvert y\rvert \geq 1$	$\{(0,0)\}$	N	N	N