Metric Spaces: Definitions and Examples- HW Problems

1. Prove from the definition of a metric space that $\mathbb{R}^2, d$ is a metric space where

$$d(p,q) = |p_1 - q_1| + |p_2 - q_2|; \qquad p = (p_1, p_2), \qquad q = (q_1, q_2).$$

2. Let A={positive Integers} and B={all Integers}. Let d be defined by: $d(p,q) = |p^2 - q^2|$.

   a. Prove that A, d is a metric space.

   b. Prove that B, d is not a metric space.

3. Prove that $d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1|$ is not a metric on $\mathbb{R}^2$.

4a. Define a metric on $\mathbb{R}$ by $d(p,q) = |e^p - e^q|$. Find all of the points $p \in \mathbb{R}$ such that $d(p, 2) < 1$.

b. Using the metric in problem number 1, find the set of all points $p = (p_1, p_2) \in \mathbb{R}^2$ such that $d(p, O) \leq 1$, where $O = (0, 0)$. Sketch this set in $\mathbb{R}^2$.

5. Let $X = C[0,1]$= the set of real valued, continuous functions on $[0,1]$. Define 2 metrics on X by

$d_1(f(x), g(x)) = \int_0^1 |f(x) - g(x)| dx$ and

$d_2(f(x), g(x)) = \max_{x \in [0,1]} |f(x) - g(x)|$.

Let $f(x) = x$ and $g(x) = x^2$.

Find $d_1(f(x), g(x))$, and $d_2(f(x), g(x))$.

6. Make a table of the subsets of $\mathbb{R}^2$ below (using the standard metric on $\mathbb{R}^2$) where across the top you have the categories: "Limit Points", "Isolated Points", "Bounded", "Closed", and "Open", and along the left side you have the sets A through I. Identify all limit points and isolated points. Put "Y" or "N" for the rest.

$A = \{(x, y) | \; x^2 + y^2 \leq 1\}$

$B = \{(x, y) | \; 0 < x^2 + y^2 \leq 1\}$

$C = \{(x, y) | \; 0 < x^2 + y^2 < 1\}$

$D = \{(x, y) | \; \frac{1}{2} < x^2 + y^2 < 1\} \cup \{(0,0)\}$

$E = \{(x, y) | \; 0 < x < 2, \quad y = 1\}$

$F = \{(x, y) | \; 0 \leq x \leq 2, \quad 0 \leq y \leq 1\}$

$G = \{(x, y) | \; 1 \leq x, \quad 1 \leq y \leq 2\}$

$H = \{(0,0), (0,1), (1,0)\}$

$I = \{(x, y) | \; y = \sin(x)\}$