Uniform Convergence

Def. Suppose $\{f_n(x)\}$ is a sequence of functions $f_n: I \subseteq \mathbb{R} \to \mathbb{R}$, where I is an interval (bounded or unbounded, open, closed, or neither) in $\mathbb{R}$. We say $\boldsymbol{\{f_n(x)\}}$ **converges pointwise to $\boldsymbol{f(x)}$,** and write $\lim_{n\to\infty} f_n(x) = f(x)$, if for each $x \in I$, the sequence of real numbers $\{f_n(x)\}$ converges to $f(x)$.

That is, for all $\epsilon > 0$ there exists an $N_x \in \mathbb{Z}^+$ such that if $n \geq N_x$ then $|f_n(x) - f(x)| < \epsilon$.

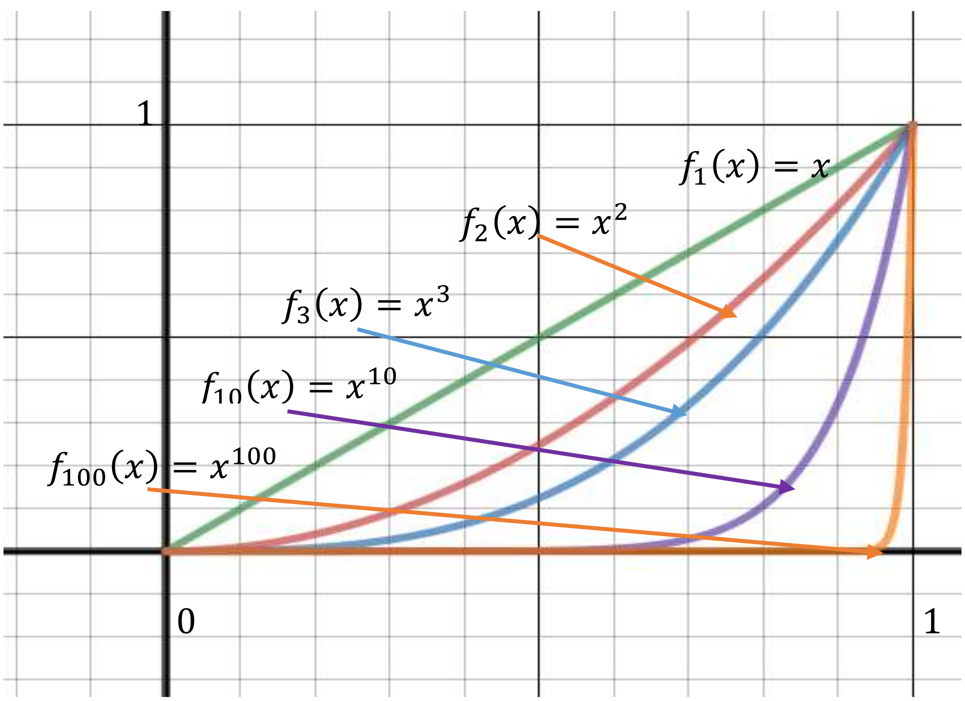
Ex. Let $f_n(x) = x^n$, on $I = [0,1]$. Prove that:

$$\lim_{n\to\infty} f_n(x) = f(x) = 0 \qquad if \quad 0 \leq x < 1$$

$$= 1 \qquad if \quad x = 1.$$

For example, if $x = \frac{1}{2}$, the sequence $\{f_n(\frac{1}{2})\} = \{\left(\frac{1}{2}\right)^n\} \to 0$ as $n \to \infty$.

However, if $x = 1$, the sequence $\{f_n(1)\} = \{(1)^n\} \to 1$ as $n \to \infty$.



We must show given any $\epsilon > 0$ there exists an $N_x \in \mathbb{Z}^+$, such that if $n \geq N_x$ then $|x^n - f(x)| < \epsilon$.

If $x = 1$, then $|1^n - 1| = 0 < \epsilon$ for any n, so we can choose $N_x = 1$.

If $x = 0$, then $|0^n - 0| = 0 < \epsilon$ for any n, so we again can choose $N_x = 1$.

If $0 < x < 1$, then: $|x^n - 0| < \epsilon$

$$|x|^n < \epsilon$$

$$(n)ln|x| < ln\epsilon$$

$$n > \frac{ln\epsilon}{\ln|x|}$$ (since $\ln|\mathrm{x}| < 0$ because $0 < x < 1$)

So choose $N_x > \max(\frac{ln\epsilon}{\ln|x|}, 0)$;

If $n \geq N_x$ then:

$$|x^n - 0| = |x|^n < |x|^{\frac{ln\epsilon}{\ln|x|}} = (e^{\ln|x|})^{\frac{ln\epsilon}{\ln|x|}} = e^{ln\epsilon} = \epsilon.$$

Notice that each $f_n(x)$ in this example is a continuous function (in fact, an infinitely differentiable function), but the sequence of functions converges pointwise to a discontinuous function.

To try to avoid having a sequence of continuous functions converging to a discontinuous function, we need a "stronger" definition of "convergence".

Def. A sequence of functions $\{f_n(x)\}$, $f_n: I \subseteq \mathbb{R} \to \mathbb{R}$, where I is an interval (bounded or unbounded, open, closed, or neither) in $\mathbb{R}$, **converges uniformly to $\boldsymbol{f(x)}$** if for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$, such that for ALL $x \in I$, if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon$.

1. Notice that for Pointwise convergence the N can depend on the point $x \in I$ as well as ϵ. For Uniform convergence the N depends only on ϵ and NOT the point $x \in I$.

2. Uniform convergence is a stronger condition than pointwise convergence. Thus if a sequence of functions converges uniformly to a function $f(x)$, then it must converge pointwise to $f(x)$. However, if a sequence of functions converges pointwise to $f(x)$ then it may, or may not, converge uniformly to $f(x)$.

Ex. Show the sequence of functions $\{x^n\}$ converges pointwise to the function:

$$f(x) = 0 \quad \text{if } 0 \leq x < 1$$

$$\quad = 1 \quad \text{if } x = 1$$

on $I = [0,1]$, but not uniformly.

In the previous example we saw that $\{x^n\}$ converges pointwise to $f(x)$. To see that any N we use must depend on the $x \in [0,1]$, notice that if $0 < x < 1$ and we try to solve for an n from the epsilon statement we get:

$$|x^n - 0| < \epsilon \text{ is equivalent to } n > \frac{ln\epsilon}{\ln|x|}$$

Thus if $\epsilon < 1$, as x goes to 1, $\frac{ln\epsilon}{\ln|x|}$ goes to ∞, thus there is no N that will work for all $0 \leq x \leq 1$.

Another way to see this is if we choose $\epsilon = \frac{1}{2}$, given any positive integer n, we can always find an x, where $0 < x < 1$ and $|x^n - 0| \geq \frac{1}{2}$.

$|x^n| \geq \frac{1}{2}$ is equivalent to $x \geq (\frac{1}{2})^{\frac{1}{n}}$ (notice that $0 < (\frac{1}{2})^{\frac{1}{n}} < 1$).

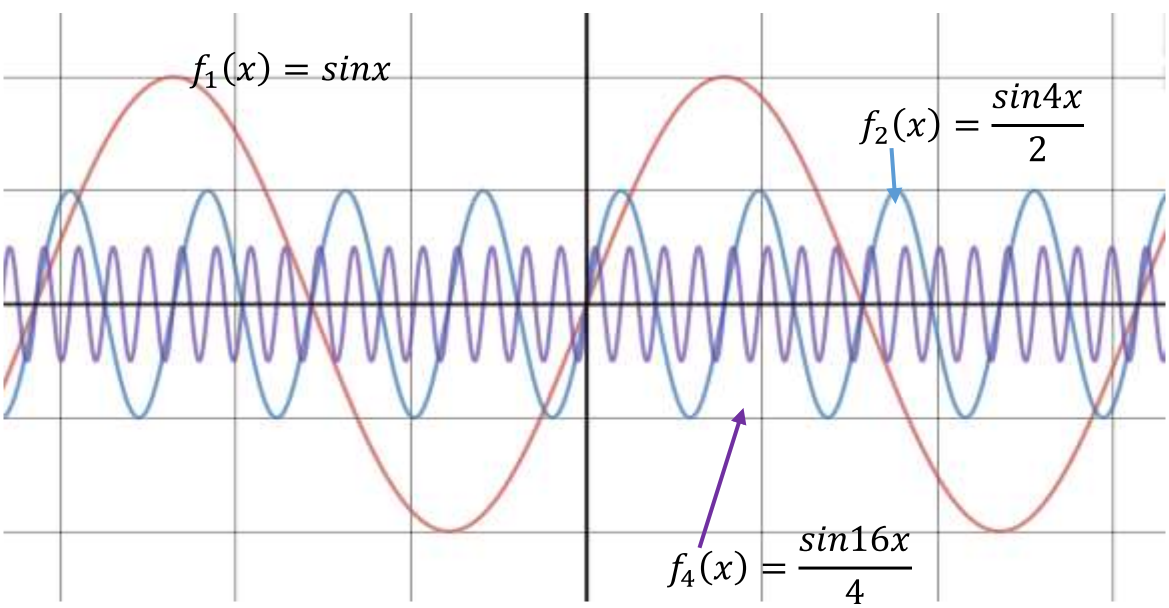
Thus for any positive integer n, $x = (\frac{1}{2})^{\frac{1}{n}}$, has $|x^n - 0| = \left|((\frac{1}{2})^{\frac{1}{n}})^n\right| = \frac{1}{2} \geq \frac{1}{2}$.

Notice that if $I = [0, \frac{7}{8}]$, $\{x^n\}$ would converge uniformly to $f(x) = 0$.

In this case we would just note that: $\frac{ln\epsilon}{\ln|x|} \leq \frac{ln\epsilon}{\ln|\frac{7}{8}|}$ for all $x \in [0, \frac{7}{8}]$.

So we could choose $N > \max(\frac{ln\epsilon}{\ln|\frac{7}{8}|}, 0)$ which does not depend on x.

Ex. Show that the sequence of functions $f_n(x) = \frac{\sin(n^2x)}{n}$ converges uniformly to $f(x) = 0$ for $I = \mathbb{R}$. However, show that $f_n{}'(x)$ does not converge even pointwise to $f'(x)$.



To show that the sequence of functions $f_n(x) = \frac{\sin(n^2x)}{n}$ converges uniformly to $f(x) = 0$ for $I = \mathbb{R}$, we must show:

for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ such that for all $x \in \mathbb{R}$, if $n \geq N$ then

$$\left|\frac{\sin(n^2x)}{n} - 0\right| < \epsilon.$$

As usual, we start with the epsilon statement:

$\left|\frac{\sin(n^2x)}{n} - 0\right| = \left|\frac{\sin(n^2x)}{n}\right| \leq \frac{1}{n};$ since $|\sin(b)| \leq 1$, for all $b \in \mathbb{R}$.

So if we can force $\frac{1}{n} < \epsilon$ we're almost done, because $\left|\frac{\sin(n^2x)}{n} - 0\right| \leq \frac{1}{n}$.

But $\frac{1}{n} < \epsilon$ is equivalent to $n > \frac{1}{\epsilon}$.

So choose $N > \frac{1}{\epsilon}$ (notice that N depends only on ϵ and not $x \in \mathbb{R}$).

If $\mathrm{n} \geq N > \frac{1}{\epsilon}$ we have:

$$\left|\frac{\sin(n^2x)}{n} - 0\right| = \left|\frac{\sin(n^2x)}{n}\right| \leq \frac{1}{n} < \frac{1}{\frac{1}{\epsilon}} = \epsilon.$$

Thus $f_n(x) = \frac{\sin(n^2x)}{n}$ converges uniformly to $f(x) = 0$ for $I = \mathbb{R}$.

Now notice that $f_n'(x) = \frac{n^2\cos(n^2x)}{n} = ncos(n^2x)$, and $f'(x) = 0$.

However, for no value of x is $\lim_{n\to\infty} f_n'(x) = 0$, in fact the $\lim_{n\to\infty} f_n'(x)$ does not exist (at least it's not a finite number). For example, when $x = 0$,

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} n = \infty.$$

Theorem: If $f_n(x)$ converges to $f(x)$ uniformly on an interval $I \subseteq \mathbb{R}$, and $f_n(x)$ is continuous on I for all n, then $f(x)$ is continuous on I.

Proof: we must show that given any point $a \in I$, that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta,\ x \in I,$ then $|f(x) - f(a)| < \epsilon$ (here the δ can depend on the point "a").

Let's start by choosing any point $a \in I$, and fixing any $\epsilon > 0$.

By the triangle inequality we know:

$$|f(x) - f(a)| \leq |f(x) - f_n(x)| + |f_n(x) - f(a)|.$$

Using the triangle inequality again, but on the 2nd term on the RHS we get:

$$|f_n(x) - f(a)| \leq |f_n(x) - f_n(a)| + |f_n(a) - f(a)|.$$

Putting these 2 triangle inequalities together we get:

$$|f(x) - f(a)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|.$$

Now let's show that each one of the terms on the RHS can be made less than $\frac{\epsilon}{3}$.

Since $f_n(x)$ converges to $f(x)$ uniformly we know there exists a $N \in \mathbb{Z}^+$ such that if $n \geq N$ then $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for every $x \in I$.

Thus the first and the third terms on the RHS can be made less than $\frac{\epsilon}{3}$ by choosing any $n \geq N$, using N in the statement above.

Since $f_n(x)$ is continuous on I we know that given any $\frac{\epsilon}{3} > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta,\ x \in I,$ then $|f_n(x) - f_n(a)| < \frac{\epsilon}{3}$.

Using this δ we have:

$$\begin{aligned} |f(x) - f(a)| &\le |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus $f(x)$ is continuous on I.

Def. Let $\boldsymbol{C(I)} = \{\text{bounded continuous functions } f : I \subseteq \mathbb{R} \to \mathbb{R}\}$

Note: If I is closed and bounded then it is compact and thus any continuous function on I will automatically be bounded.

$C(I)$ is a metric space with the distance defined as:

$$d\big(f(x), g(x)\big) = \operatorname*{Sup}_{x \in I} |f(x) - g(x)|$$

1. $d\big(f(x), g(x)\big) = \operatorname*{Sup}_{x \in I} |f(x) - g(x)| \ge 0$; and $d\big(f(x), g(x)\big) = 0$ implies $f(x) = g(x)$.
2. $d\big(f(x), g(x)\big) = d\big(g(x), f(x)\big)$
3. $d\big(f(x), g(x)\big) \le d\big(f(x), h(x)\big) + d\big(h(x), g(x)\big)$
   This is true because if $A(x) = B(x) + E(x)$ then by the triangle inequality:
   $$|A(x)| \le |B(x)| + |E(x)| \text{ for any } x \in I.$$
   Thus we have: $\operatorname*{Sup}_{x \in I}|A(x)| \le \operatorname*{Sup}_{x \in I}|B(x)| + \operatorname*{Sup}_{x \in I}|E(x)|$.

Now let $A(x) = f(x) - g(x),\quad B(x) = f(x) - h(x),\quad E(x) = h(x) - g(x)$.

This gives us $d\big(f(x), g(x)\big) \le d\big(f(x), h(x)\big) + d\big(h(x), g(x)\big)$.

Notice that a sequence of functions $f_n(x) \in C(I)$ converges to $f(x)$ with this metric if given any $\epsilon > 0$ there exists a $N \in \mathbb{Z}^+$ such that if $n \geq N$ then $d\big(f_n(x), f(x)\big) = \operatorname*{Sup}_{x \in I} |f_n(x) - f(x)| < \epsilon$.

This ϵ statement is equivalent to saying that $|f_n(x) - f(x)| < \epsilon$ for all $x \in I$. Thus convergence in $C(I)$ is the same as uniform convergence.

We already know that if $f_n(x)$ converges uniformly to $f(x)$ and all of the $f_n(x)$ are continuous then so is $f(x)$. In a moment we'll see that if all of the $f_n(x)$ are also bounded, then so is $f(x)$. Thus any sequence in $C(I)$ that "converges" with the above metric, converges to a function in $C(I)$. Thus $C(I)$ is a complete metric space.

Theorem: $f_n(x)$ converges uniformly to $f(x)$ on I if and only if for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$, such that for all $x \in I$, if $n, m \geq N$ then $|f_n(x) - f_m(x)| < \epsilon$.

(i.e., if $\{f_n(x)\} \subseteq C(I)$, then $\{f_n(x)\}$ converges to $f(x) \in C(I)$, if and only if $\{f_n(x)\}$ is a Cauchy sequence in $C(I)$).

Proof: Assume that $f_n(x)$ converges uniformly to $f(x)$ on I.

By the triangle inequality we have:

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)|.$$

Since $f_n(x)$ converges uniformly to $f(x)$ on I, there exists $N \in \mathbb{Z}^+$ such that if $n \geq N$ then $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for any $x \in I$.

And, of course, if $m \geq N$ then $|f_m(x) - f(x)| < \frac{\epsilon}{2}$ for any $x \in I$.

Thus if $m, n \geq N$ then we have:

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for any $x \in I$.

Now assume for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$, such that for all $x \in I$, if $n, m \geq N$ then $|f_n(x) - f_m(x)| < \epsilon$ and show that $f_n(x)$ converges uniformly to $f(x)$ on I.

For each $x \in I$, $\{f_n(x)\}$ is a Cauchy sequence of real numbers and thus converges to a real number $f(x)$. So $\lim_{n\to\infty} f_n(x) = f(x)$ (this is a pointwise limit).

Now we must show that $f_n(x)$ converges uniformly to $f(x)$.

By assumption, there exists an $N \in \mathbb{Z}^+$, such that for all $x \in I$, if $n, m \geq N$ then $|f_n(x) - f_m(x)| < \epsilon$.

This is true for all $m \geq N$, so let m go to ∞. Thus we have:

there exists an $N \in \mathbb{Z}^+$, such that for all $x \in I$, if $n \geq N$

then $|f_n(x) - f(x)| < \epsilon$.

Hence $f_n(x)$ converges to $f(x)$ uniformly.

Now we can see why a set of bounded uniformly convergent continuous functions must converge to a bounded continuous function. Suppose $|f_n(x)| \leq M_n$ for all $x \in I$ and each n. How do we know that as n goes to infinity, M_n doesn't go to infinity?

By the previous theorem we know that any Cauchy sequence in $C(I)$, $\{f_n(x)\}$, converges to uniformly to some $f(x)$ on I (which must be continuous since all of the $f_n's$ are). Thus we have for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$, such that for all $x \in I$, if $n \geq N$ then $|f(x) - f_n(x)| < \epsilon$.

In particular, $|f(x) - f_N(x)| < \epsilon$ for all $x \in I$. Thus we have:

$$-\epsilon < f(x) - f_N(x) < \epsilon$$

$$f_N(x) - \epsilon < f(x) < f_N(x) + \epsilon$$

$$-M_N - \epsilon \leq f_N(x) - \epsilon < f(x) < f_N(x) + \epsilon \leq M_N + \epsilon$$

Thus $|f(x)| \leq M_N + \epsilon$ and $f(x)$ is bounded.

Hence any Cauchy sequence in $C(I)$ must converge to a bounded continuous function, $f(x)$, thus $f(x) \in C(I)$ and $C(I)$ is complete.