Uniform Convergence- HW Problems

1. Let $\{f_n(x)\}$ be defined by:

$$f_n(x) = \sqrt[n]{x}\,; \quad \text{for } 0 \leq x \leq 1.$$

   a. Prove with an ϵ-N argument that $\{f_n(x)\}$ converges pointwise to the function:
   $$f(x) = 1 \quad if \quad 0 < x \leq 1$$
   $$= 0 \quad if \quad x = 0.$$

   b. Show that $\{f_n(x)\}$ does not converge uniformly to $f(x)$ by showing that given any $n > 0$ you can find a point $0 < x < 1$ such that $\left|\sqrt[n]{x} - 1\right| \geq \frac{1}{2}$.
   (Note: the point x can be a function of n)

   c. Using the ϵ-N definition of uniform convergence, prove that $\{f_n(x)\}$ converges uniformly to: $f(x) = 1,$ on the interval $[\frac{1}{2}, 1]$.

2. Suppose that $\{f_n(x)\}$ and $\{g_n(x)\}$ are sequences of real-valued functions on $\mathbb{R}$ such that $\{f_n(x)\}$ converges uniformly to $f(x)$ on $\mathbb{R}$ and $\{g_n(x)\}$ converges uniformly to $g(x)$ on $\mathbb{R}$. Prove with an ϵ-N argument that $\{f_n(x) + g_n(x)\}$ converges uniformly to $f(x) + g(x)$ on $\mathbb{R}$.