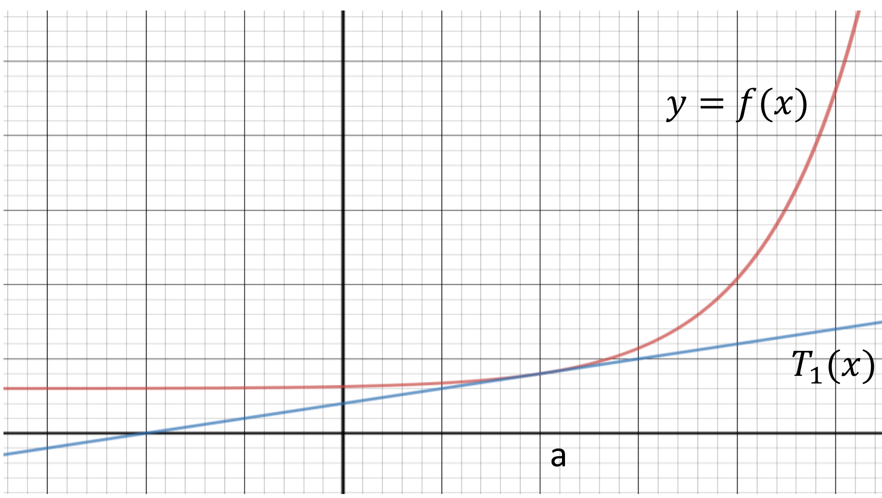
Taylor Series

Starting with a function $f(x)$ which has infinitely many derivatives we can form a Taylor Polynomial of degree n about a point $x = a$.

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n .$$

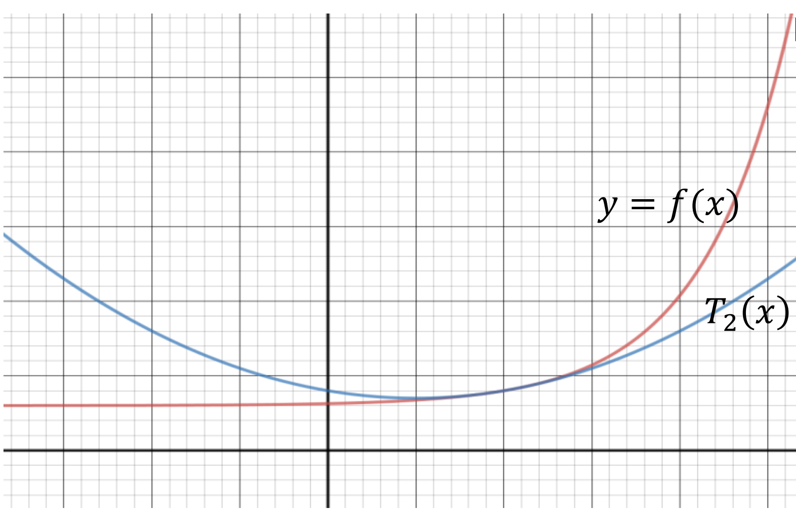
$T_1(x) = f(a) + f'(a)(x-a)$ is a linear approximation of $f(x)$.



Here we have: $T_1(a) = f(a)$

$T_1{}'(a) = f'(a)$

$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$ is a quadratic approximation of f.



Here we have: $T_2(a) = f(a)$

$T_2{}'(a) = f'(a)$

$T_2{}''(a) = f''(a)$

$T_n(x)$ is an approximation of the function $f(x)$ which has:

$T_n(a) = f(a)$, $T_n{}'(a) = f'(a)$, $T_n{}''(a) = f''(a)$, …, $T_n{}^{(n)}(a) = f^{(n)}(a)$.

The question is, how "good" an approximation is $T_n(x)$ of $f(x)$ when $x \neq a$?

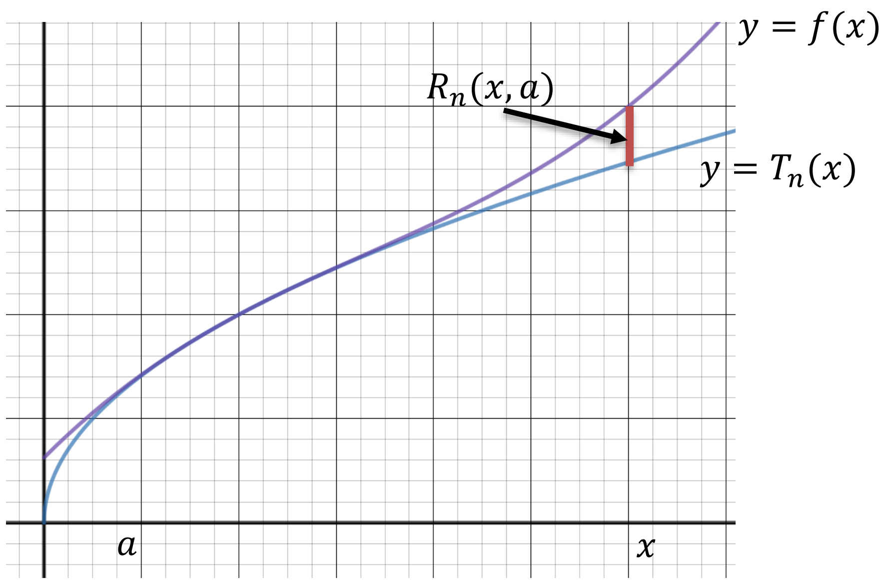
Can we put some kind of bound on how large the error is?

Theorem (Taylor's Formula); If f has $n+1$ derivatives in an interval I that contains "a" , then for $x \epsilon I$ there is a number c, where c is strictly between x and a, such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$+ \frac{f^n(a)}{n!}(x-a)^n + R_n(x,a).$$

where the error after the nth degree term, $R_n(x,a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$.



Note 1: "c" depends on x and a.

Note 2: When $n = 0$ we have:

$$f(x) = f(a) + f'(c)(x - a) \quad \text{or} \quad \frac{f(x)-f(a)}{x-a} = f'(c);$$

with c between x and a, which is just the Mean Value Theorem.

Note 3: Taylor's formula is important because it allows us to explicitly estimate how big the error is.

Proof: We will create a function that satisfies the Mean Value Theorem and the expression $R_n(x,a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, will follow from the M.V.T.

Let's start by fixing x and a, i.e. x and a are now constants where $x \neq a$.

We define a function $g(t)$ by:

$$g(t) = f(x) - [f(t) + f'(t)(x-t) + \frac{f''(t)}{2!}(x-t)^2 + \cdots + \frac{f^n(t)}{n!}(x-t)^n + R_n(x,a)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}]$$

Notice that:

$$g(x) = f(x) - [f(x) + f'(x)(x-x) + \frac{f''(x)}{2!}(x-x)^2 + \cdots + \frac{f^n(x)}{n!}(x-x)^n + R_n(x,a)\frac{(x-x)^{n+1}}{(x-a)^{n+1}}] = 0$$

$$g(a) = f(x) - [f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_n(x,a)\frac{(x-a)^{n+1}}{(x-a)^{n+1}}]$$

$$= f(x) - f(x) = 0.$$

$g(t)$ satisfies the Mean Value Theorem on an interval containing $x\ and\ a.$

So by the Mean Value Theorem, there exists a c between a and x such that:

$$\frac{g(x)-g(a)}{x-a} = 0 = g'(c).$$

$$g'(t) = -[f'(t) - f'(t) + f''(t)(x-t) + \frac{1}{2!}\left[(-2(x-t)f''(t) + (x-t)^2 f'''(t)] + \cdots + \frac{f^{(n+1)}(t)}{n!}(x-t)^n - (n+1)R_n(x,a)\frac{(x-t)^n}{(x-a)^{n+1}}\right].$$

Which simplifies to:

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x,a)\frac{(x-t)^n}{(x-a)^{n+1}}\,.$$

Since $g'(c) = 0$ we have:

$$0 = g'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x,a)\frac{(x-c)^n}{(x-a)^{n+1}}\,.$$

Solving for $R_n(x,a)$ we get:

$$\frac{f^{(n+1)}(c)}{n!}(x-c)^n = (n+1)R_n(x,a)\frac{(x-c)^n}{(x-a)^{n+1}}$$

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^n = R_n(x,a)\frac{(x-c)^n}{(x-a)^{n+1}}$$

$\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} = R_n(x,a)\,;$ where c is between x and a.

Ex. Compute the Talylor polynomial $T_3(x)$ around $a = 0$ for $f(x) = sinx$ and use it to estimate $sin(0.1)$. Find a bound for the error in this estimate.

$$T_3(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3$$

$f(x) = sinx$ $\qquad$ $f(0) = sin0 = 0$

$f'(x) = cosx$ $\qquad$ $f'(0) = cos0 = 1$

$f''(x) = -sinx$ $\qquad$ $f''(0) = -sin0 = 0$

$f'''(x) = -cosx$ $\qquad$ $f'''(0) = cos0 = -1$

$f^4(x) = sinx$

$$T_3(x) = x - \frac{x^3}{3!}.$$

$$f(x) \approx T_3(x) = x - \frac{x^3}{3!}$$

$$f(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} \approx 0.099833$$

$$f(x) = sinx = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + R_3(x,0);$$

$R_3(x,0) = \frac{f^4(c)}{(4)!}(x)^4 = \frac{sinc}{(4)!}(x)^4$; where c is between 0 and x.

$$|R_3(0.1,0)| = |\frac{sinc}{(4)!}(0.1)^4| \le \frac{1}{24}(0.1)^4 \approx 0.000004.$$

This means that :

$$0.099833 - 0.000004 < \sin(0.1) < 0.099833 + 0.000004 \quad \text{or}$$

$$0.099829 < \sin(0.1) < 0.099837\ .$$

Ex. Approximate $ln(1.2)$ so that the error is less than 0.001.

Find the Taylor series with error term for $f(x) = lnx$ around the point $a = 1$.

$f(x) = lnx$ $\qquad f(1) = ln1 = 0$

$f'(x) = \frac{1}{x}$ $\qquad f'(1) = \frac{1}{1} = 1$

$f''(x) = \frac{-1}{x^2}$ $\qquad f''(1) = \frac{-1}{(1)^2} = -1$

$f'''(x) = \frac{2}{x^3}$ $\qquad f'''(1) = \frac{2}{(1)^3} = 2$

$f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{x^n}$ $\qquad f^{(n)}(1) = (-1)^{n+1}\frac{(n-1)!}{1^n} = (-1)^{n+1}(n-1)!$

$$T_n(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \ldots + \frac{f^n(1)}{n!}(x-1)^n\ .$$

$$R_n(x,1) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1} = (-1)^{n+2}\frac{(n!)}{c^{n+1}}\left(\frac{1}{(n+1)!}\right)(x-1)^{n+1}$$

$$= (-1)^{n+2}\frac{(x-1)^{n+1}}{(n+1)c^{n+1}}\ .$$

$|R_n(x,1)| = |\frac{(x-1)^{n+1}}{(n+1)c^{n+1}}|$ where c is between 1 and x.

Now $x = 1.2$ so:

$$|R_n(1.2,1)| = \left|\frac{(1.2-1)^{n+1}}{(n+1)c^{n+1}}\right| = \frac{(0.2)^{n+1}}{(n+1)c^{n+1}}.$$

Since c is between 1 and 1.2:

$|R_n(1.2,1)| < \frac{(0.2)^{n+1}}{(n+1)}$ and we want this to be less than 0.001.

So we must solve for n: $\frac{(0.2)^{n+1}}{(n+1)} < 0.001.$

There's no elementary way to do this, but we can just use trial and error. Just try $n = 1, 2, 3, \ldots$ until we find an n that works. $n = 3$ will do the trick.

Thus we can say:

$$\ln(1.2) \approx f(1) + f'(1)(1.2-1) + \frac{f''(1)}{2!}(1.2-1)^2 + \frac{f'''(1)}{3!}(1.2-1)^3$$

with an error less than 0.001.

$$\ln(1.2) \approx 0 + (1.2-1) - \frac{1}{2}(1-1.2)^2 + \frac{2}{6}(1-1.2)^3 \approx 0.1827$$

with an error less than 0.001.

So we know that :

$$0.1827 - 0.001 < \ln(1.2) < 0.1827 + 0.001$$

$$0.1817 < \ln(1.2) < 0.1837.$$

Ex. Find the negative values for x where $f(x) = e^x$ can be approximated by $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ with an error less than 0.001.

$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ around $a = 0$.

$R_3(x,0) = \frac{f^{(4)}(c)}{(4)!}(x)^4 = \frac{e^c}{24}x^4$ we want to know the x's such that $x < 0$ and $|R_3(x,0)| < 0.001$.

Since $x < 0$ and $a = 0$, c, which is between x and a, is also less than 0. Thus $e^c < 1$.

$|R_3(x,0)| = \left|\frac{e^c}{24}x^4\right| < \left|\frac{x^4}{24}\right| < 0.001$; Now let's solve $x^4 < 0.024$

$|x| < \sqrt[4]{0.024} \approx 0.3936$

So $-0.3936 < x < 0$.

Suppose $f(x)$ has infinitely many derivatives for $x \epsilon \mathbb{R}$. When is $f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$? That is, when does the Taylor Series of a function converge to the values of the function?

Theorem: If $\lim_{n\to\infty} R_n(x,a) = 0$ for $|x-a| < M$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n \text{ for all } x \text{ such that } |x-a| < M.$$

Proof: $f(x) = T_n(x) + R_n(x,a)$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$+ \frac{f^n(a)}{n!}(x-a)^n + R_n(x,a).$$

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} [\sum_{i=0}^{n} \frac{f^i(a)}{i!}(x-a)^i + R_n(x,a)]$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n \ ; \quad \text{since } \lim_{n\to\infty} R_n(x,a) = 0.$$

Ex. Prove that the Taylor Series around $a = 0$ for $f(x) = e^x$ converges to $f(x) = e^x$ for all $x \epsilon \mathbb{R}$.

$f(x) = e^x$ $\qquad f(0) = e^0 = 1$

$f'(x) = e^x$ $\qquad f'(0) = 1$

$f''(x) = e^x$ $\qquad f''(0) = 1$

$f^{(n)}(x) = e^x$ $\qquad f^{(n)}(0) = 1$

$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

$|R_n(x,0)| = |\frac{f^{(n+1)}(c)}{(n+1)!}(x)^{n+1}| = |\frac{e^c}{(n+1)!}(x)^{n+1}|$;

where c is between 0 and x.

We need to show that $\lim_{n\to\infty} |\frac{e^c}{(n+1)!}(x)^{n+1}| = 0$ for any $x \epsilon \mathbb{R}$.

Thus we just have to show for any fixed number x, $\lim_{n\to\infty} \frac{|(x)|^{n+1}}{(n+1)!} = 0$, since e^c is just a constant once x is fixed.

Fix x and let $p = [|x|]$ =the greatest integer less than or equal to $|x|$.

Notice that:

$$\frac{|x|^n}{n!} = \left(\frac{|x|}{1}\right)\left(\frac{|x|}{2}\right)\ldots\left(\frac{|x|}{p}\right)\left(\frac{|x|}{p+1}\right)\ldots\left(\frac{|x|}{n}\right) \le \left(\frac{|x|^p}{p!}\right)\left(\frac{|x|}{(p+1)}\right)^{n-p}; \text{ where } \frac{|x|}{(p+1)} < 1.$$

We now just need to show that $\lim_{n\to\infty} \left(\frac{|x|}{(p+1)}\right)^{n-p} = 0$ since

$$0 \le \frac{|x|^n}{n!} \le \left(\frac{|x|^p}{p!}\right)\left(\frac{|x|}{(p+1)}\right)^{n-p}.$$

Thus by the squeeze theorem if $\lim_{n\to\infty} \left(\frac{|x|^p}{p!}\right)\left(\frac{|x|}{(p+1)}\right)^{n-p} = 0$ then $\lim_{n\to\infty} \frac{|(x)|^n}{(n)!} = 0$.

Since $\frac{|x|}{(p+1)} < 1$, and $\frac{|x|^p}{p!}$ is a constant, if we can just show that $\lim_{n\to\infty} \alpha^n = 0$ if $|\alpha| < 1$, we will be done.

We must show given any $\epsilon > 0$ we can find $N > 0$ such that if $n \geq N$ then $|\alpha^n - 0| < \epsilon$.

$$|\alpha|^n < \epsilon$$

$$n(\ln|\alpha|) < \ln(\epsilon)$$

$$n > \frac{\ln(\epsilon)}{\ln|\alpha|} \qquad \text{since } \ln|\alpha| < 0.$$

If $\epsilon > 1$ then $\frac{\ln(\epsilon)}{\ln|\alpha|} < 0$ so let's choose $N > max(0, \frac{\ln(\epsilon)}{\ln|\alpha|})$.

Now let's show this N works.

If $n \geq N > \max(0, \frac{\ln(\epsilon)}{\ln|\alpha|})$ then

$$|\alpha^n - 0| = |\alpha|^n < |\alpha|^{\frac{\ln(\epsilon)}{\ln|\alpha|}} = (e^{\ln|\alpha|})^{\frac{\ln(\epsilon)}{\ln|\alpha|}}$$

$$= e^{\ln|\epsilon|} = \epsilon.$$

So $\lim_{n\to\infty} \frac{(|x|)^{n+1}}{(n+1)!} = 0$ and $\lim_{n\to\infty} R_n(x, 0) = 0$.

So $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Ex. Prove that the Taylor Series for $f(x) = sinx$ around $a = 0$ converges to $f(x) = sinx$ for all $x\epsilon\mathbb{R}$.

$$f(x) = sinx \qquad f(0) = sin0 = 0$$

$$f'(x) = cosx \qquad f'(0) = cos0 = 1$$

$$f''(x) = -sinx \qquad f''(0) = -sin0 = 0$$

$$f^{(3)}(x) = -cosx \qquad f^{(3)}(0) = -cos0 = -1$$

$$f^{(4)}(x) = sinx \qquad f^{(4)}(0) = sin0 = 0.$$

$$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$R_{2n+1}(x,0) = \frac{f^{(2n+2)}(c)}{(2n+2)!}(x)^{2n+2} .$$

We must show that $\lim_{n\to\infty} R_{2n+1}(x,0) = 0$, for all $x\epsilon\mathbb{R}$.

$$|R_{2n+1}(x,0)| = \left|\frac{f^{(2n+2)}(c)}{(2n+2)!}(x)^{2n+2}\right|.$$

Notice that all of the derivatives of $f(x) = sinx$ are $\pm sinx$ or $\pm cosx$. In any of those cases we know that $|f^{(n)}(x)| \leq 1$, for all x. Thus we know:

$|R_{2n+1}(x,0)| = |\frac{f^{(2n+2)}(c)}{(2n+2)!}(x)^{2n+2}| \leq |\frac{x^{2n+2}}{(2n+2)!}|$. But notice that:

$0 \leq |R_{2n+1}(x,0)| \leq |\frac{x^{2n+2}}{(2n+2)!}|$ and we just saw that $\lim_{n\to\infty} \frac{|(x)|^n}{(n)!} = 0.$

So by the squeeze theorem $\lim_{n\to\infty} R_{2n+1}(x,0) = 0$, for all $x\epsilon\mathbb{R}$.

Thus the Taylor Series for $f(x) = sinx$ converges to $f(x) = sinx$ for all $x\epsilon\mathbb{R}$.

Ex. Let $f(x) = \ln(1-x)$.

a. Find $T_3(x)$ around a=0.

b. Approximate $ln(1.2)$ using $T_3(x)$ around $a = 0$.

c. Find an upper bound for the error in the approximation in part b.

d. Prove the Taylor series for $\ln(1-x)$, $-\sum_{n=1}^{\infty} \frac{x^n}{n}$, converges to $\ln(1-x)$ for $-1 < x < 0$.

a. $T_3(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3$ (around $a = 0$)

$f(x) = \ln(1-x)$ $\qquad$ $f(0) = ln1 = 0$

$f'(x) = -\frac{1}{1-x}$ $\qquad$ $f'(0) = -1$

$f''(x) = -\frac{1}{(1-x)^2}$ $\qquad$ $f''(0) = -1$

$f'''(x) = -\frac{2}{(1-x)^3}$ $\qquad$ $f'''(0) = -2$

So we have:

$$T_3(x) = 0 - x - \frac{x^2}{2} - \frac{x^3}{3} = -x - \frac{x^2}{2} - \frac{x^3}{3}.$$

b. $\ln(1-x) \approx -x - \frac{x^2}{2} - \frac{x^3}{3}$; so at $x = -0.2$ we have:

$$\ln(1.2) \approx -(-0.2) - \frac{(-0.2)^2}{2} - \frac{(-0.2)^3}{3} \approx 0.1827.$$

c. The remainder, or error term, for $T_n(x)$ around $a = 0$ is:

$R_n(x,0) = \frac{f^{(n+1)}(c)}{(n+1)!}(x)^{n+1}$. In this case, $n = 3$.

$R_3(x,0) = \frac{f^{(4)}(c)}{(4)!}(x)^4$; where c is between x and 0.

Thus the error at $x = -0.2$ is:

$R_3(-0.2,0) = \frac{f^{(4)}(c)}{(4)!}(-0.2)^4$; where $-0.2 < c < 0$.

$f^{(4)}(x) = -\frac{6}{(1-x)^4}$; so $f^{(4)}(c) = -\frac{6}{(1-c)^4}$

$R_3(-0.2,0) = \frac{1}{(4)!}(-\frac{6}{(1-c)^4})(-0.2)^4$; where $-0.2 < c < 0$

$|R_3(-0.2,0)| = |\frac{1}{(4)!}(-\frac{6}{(1-c)^4})(-0.2)^4| = \frac{(0.2)^4}{4(1-c)^4}$; $-0.2 < c < 0$.

Since $c < 0$, $1 - c > 1$; so we can say:

$|R_3(-0.2,0)| = \frac{(0.2)^4}{4(1-c)^4} < \frac{(0.2)^4}{4} = 0.0004$.

d. To prove the Taylor series converges to the function for all $-1 < x < 0$ we must show that $\lim_{n\to\infty} R_n(x,0) = 0$ for all $-1 < x < 0$.

$R_n(x,0) = \frac{f^{(n+1)}(c)}{(n+1)!}(x)^{n+1}$; where c is between x and 0.

$f^{(n+1)}(x) = -\frac{n!}{(1-x)^{n+1}}$; therefore: $f^{(n+1)}(c) = -\frac{n!}{(1-c)^{n+1}}$.

Thus we have:

$$R_n(x,0) = \frac{-1}{(n+1)!}\left(\frac{n!}{(1-c)^{n+1}}\right)(x)^{n+1} = \frac{-1}{(n+1)}\left(\frac{1}{(1-c)^{n+1}}\right)(x)^{n+1}.$$

Notice that we can rewrite this as:

$$R_n(x,0) = \frac{-1}{(n+1)}\left(\frac{x}{1-c}\right)^{n+1};$$

$$0 \le |R_n(x,0)| = \frac{1}{(n+1)}\left|\left(\frac{x}{1-c}\right)\right|^{n+1} .$$

Since $-1 < x < 0$ and $1 - c > 1$; we have $\left|\frac{x}{1-c}\right| < 1.$

If we let $\alpha = \left|\frac{x}{1-c}\right| < 1,$ we know that $\lim_{n\to\infty} \alpha^n = 0.$

Thus by the squeeze theorem we can conclude that $\lim_{n\to\infty} |R_n(x,0)| = 0$ and thus $\lim_{n\to\infty} R_n(x,0) = 0$ for all $-1 < x < 0.$

Ex. Let $f(x) = e^{-(\frac{1}{x^2})}$ for $x > 0$

$= 0$ for $x \leq 0$

Show that the Taylor Series for $f(x)$ around $a = 0$ does not converge to $f(x)$ for $x > 0$.

As we noted in the last example in the section on Differentiation, for this function $f(0) = f'(0) = \cdots = f^n(0) = 0$. Thus the Taylor series around $a = 0$ is:

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{n!}f^n(0)x^n + \cdots$$

$$= 0.$$

Clearly, $f(x) = e^{-(\frac{1}{x^2})}$ for $x > 0$

$= 0$ for $x \leq 0$

Is not equal to 0 for $x > 0$. So the Taylor series does not converge to the function for $x > 0$ even though the function has an infinite number of derivatives.