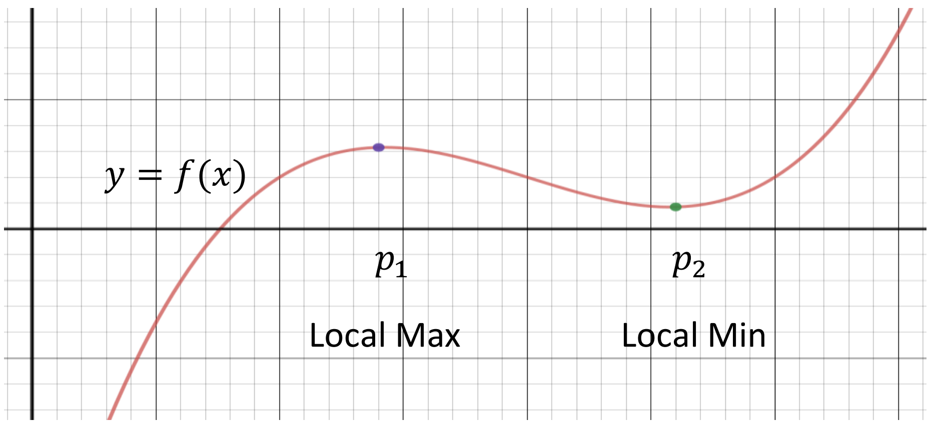
# The Mean Value Theorem

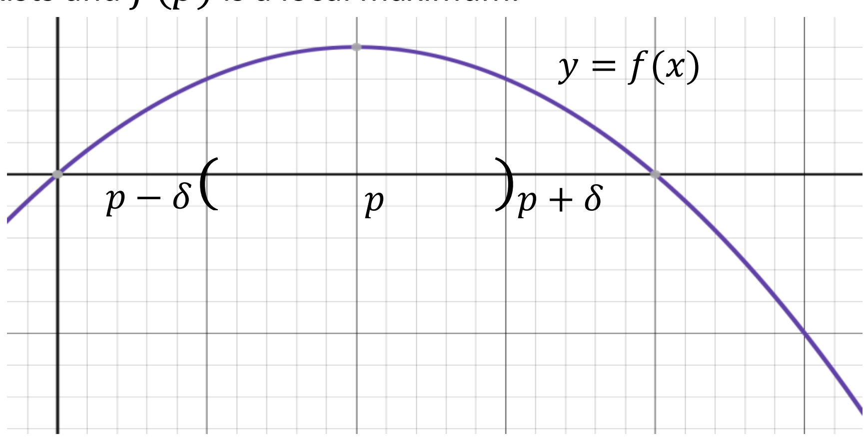
Def. Let $f$ be a real valued function defined on a metric space $X$. We say that $f$ has a **local maximum** at a point $p\epsilon X$ if there exists a $\delta > 0$ such that $f(q) \leq f(p)$ for all $q\epsilon X$ with $d_X(p,q) < \delta$.

We say that $f$ has a **local minimum** at a point $p\epsilon X$ if there exists a $\delta > 0$ such that $f(p) \leq f(q)$ for all $q\epsilon X$ with $d_X(p,q) < \delta$.




Theorem: Let $f: [a,b] \to \mathbb{R}$. If $f$ has a local maximum or minimum at a point $p\epsilon(a,b)$, and if $f'(p)$ exists, then $f'(p) = 0$.

Proof: Suppose $f'(p)$ exists and $f(p)$ is a local maximum.




Then by the definition of a local maximum, there exists a $\delta > 0$ such that

$f(x) \leq f(p)$ for all $x\epsilon[a,b]$ with $|x - p| < \delta$.

$$|x - p| < \delta$$

$$-\delta < x - p < \delta$$

$$p - \delta < x < p + \delta.$$

Suppose that we take a point $t,\ p - \delta < t < p$ , then we have:

$f(t) - f(p) \le 0$ since $f(x) \le f(p)$ for all $x \epsilon [a, b]$ with $|x - p| < \delta$ and

$t - p < 0$ since $t < p$;

So we have:

$\frac{f(t)-f(p)}{t-p} \ge 0$; for $t < p$. Thus we can say: $\lim_{t \to p^-} \frac{f(t)-f(p)}{t-p} \ge 0$ .

Now suppose we take a point , $p < t < p + \delta$.

So we have:

$f(t) - f(p) \le 0$ since $f(x) \le f(p)$ for all $x \epsilon X$ with $|x - p| < \delta$

$t - p > 0$ since $p < t$;

So we have:

$\frac{f(t)-f(p)}{t-p} \le 0$; for $t > p$. That gives us: $\lim_{t \to p^+} \frac{f(t)-f(p)}{t-p} \le 0$.

Since $f'(p)$ exists we must have:

$$0 \le \lim_{t \to p^-} \frac{f(t)-f(p)}{t-p} = \lim_{t \to p^+} \frac{f(t)-f(p)}{t-p} \le 0$$

Thus $f'(p) = \lim_{t \to p} \frac{f(t)-f(p)}{t-p} = 0$.

A similar argument works when $p$ is a local minimum.

The next theorem will be used later to prove L'Hopital's rule.

Theorem (Generalized Mean Value Theorem): If $f, g: [a, b] \to \mathbb{R}$, are continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists a point $c \epsilon (a, b)$ at which:

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

Note: we could also write this result as:

$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ ; This in turn could be written as:

$\frac{\frac{f(b)-f(a)}{b-a}}{\frac{g(b)-g(a)}{b-a}} = \frac{f'(c)}{g'(c)}$ i.e.

$$\frac{\text{the average rate of change of } f \text{ over } [a,b]}{\text{the average rate of change of } g \text{ over } [a,b]} = \frac{\text{Inst. rate of change of } f \text{ at } c}{\text{Inst. rate of change of } g \text{ at } c}.$$

Proof: Let $h(x)$ be defined by:

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x); \quad a \le x \le b.$$

$h(x)$ is continuous on $[a, b]$ and differentiable on $(a, b)$ because $f(x)$ and $g(x)$ are.

Notice that $h(a) = h(b)$:

$$h(a) = [f(b) - f(a)]g(a) - [g(b) - g(a)]f(a)$$
$$= f(b)g(a) - g(b)f(a)$$
$$h(b) = [f(b) - f(a)]g(b) - [g(b) - g(a)]f(b)$$
$$= -g(b)f(a) + f(b)g(a).$$

If we can find a point $c\epsilon(a, b)$ where $h'(c) = 0$ then we would have:

$0 = h'(c) = [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c)$ or

$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$ (which is what we are proving).

So let's show we can find a point $c\epsilon(a, b)$ where $h'(c) = 0$.

If $h(x)$ is a constant function then $h'(x) = 0$ for all $x\epsilon(a, b)$.

If $h(x)$ is not a constant function then there is some point $p$, $a < p < b$ where either $h(p) > h(a)$ or $h(p) < h(a)$.

If $h(p) > h(a)$, let $c$ be a point where $h$ attains its global maximum (we know a continuous function on a compact set attains its absolute maximum and minimum values), $a < c < b$. This global maximum is also a local maximum because it's an interior point. Thus we know from the previous theorem that since $h(x)$ is differentiable, that $h'(c) = 0$.

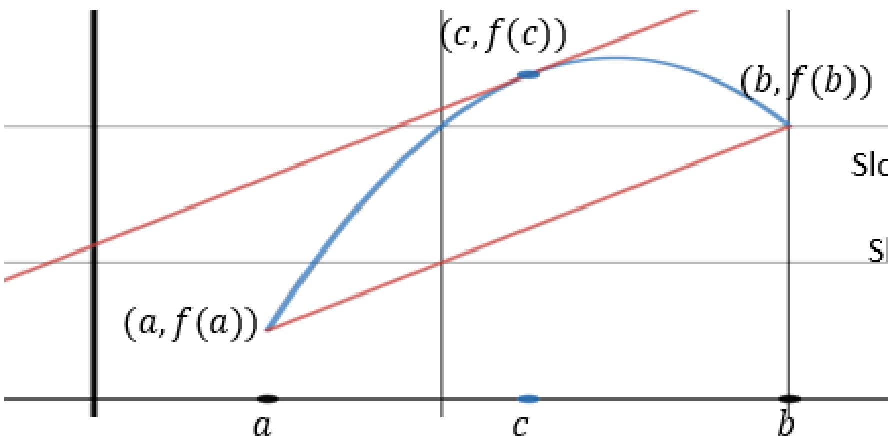
Thus at this point c we have:

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

A similar argument works for $h(p) < h(a)$.

The Mean Value Theorem: Let $f: [a, b] \to \mathbb{R}$, be continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists a $c$, $a < c < b$ such that:

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$




Slope of secant line= $\frac{f(b)-f(a)}{b-a}$

Slope of tangent line= $f'(c)$.

Proof: Let $g(x) = x$ in the generalized mean value theorem. Then we have:

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

$$[f(b) - f(a)](1) = (b - a)f'(c)$$

$$\frac{f(b)-f(a)}{b-a} = f'(c) \ ; \quad a < c < b.$$

Note: The Mean Value Theorem gives us a way to bound $|f(b) - f(a)|$ for a function:

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$f(b) - f(a) = f'(c)(b - a)$$

$$|f(b) - f(a)| = |f'(c)||(b - a)|$$

$$inf_{a<x<b}|f'(x)||b - a| \leq |f(b) - f(a)| \leq sup_{a<x<b}|f'(x)||b - a|.$$

Ex. Prove $|\sin(b) - \sin(a)| \leq |b - a|$ for all real values of $a, b$.

Apply the M.V.T. to $[a, b]$, for any $a, b$, and the function $f(x) = \sin(x)$.

$f(x) = \sin(x)$ is continuous on $[a, b]$ because it's continuous everywhere. It's differentiable on $(a, b)$ because it's differentiable everywhere.

By the mean value theorem there exists a $c$, $a < c < b$ such that:

$\frac{\sin(b)-\sin(a)}{b-a} = f'(c) \quad = \cos(c)$ or we can write:

$\sin(b) - \sin(a) = (\cos(c))(b - a)$.

Now take absolute values:

$|\sin(b) - \sin(a)| = |\cos(c)||b - a|$ ; now use the fact that $|cosc| \leq 1$

$|\sin(b) - \sin(a)| = |\cos(c)||b - a| \leq |b - a|$ so we have:

$|\sin(b) - \sin(a)| \leq |b - a|$ for all real values of $a, b$.

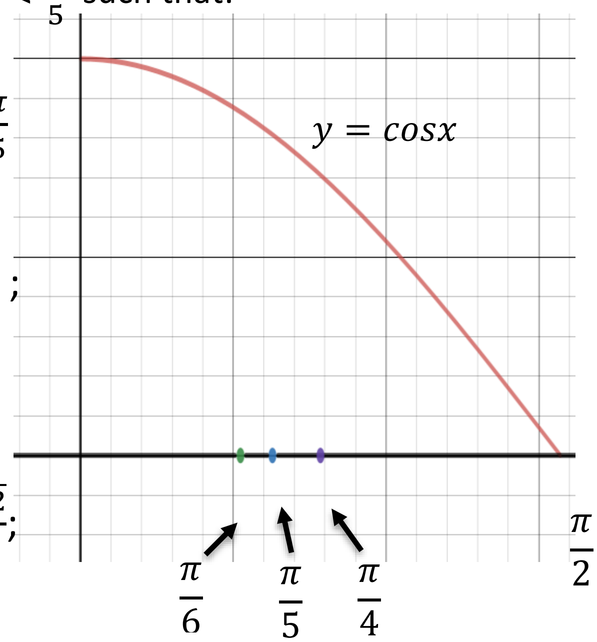
Ex. Use the M. V. Theorem to prove that $\frac{1}{2} + \left(\frac{\sqrt{2}}{60}\right)\pi < \sin\left(\frac{\pi}{5}\right) < \frac{1}{2} + (\frac{\sqrt{3}}{60})\pi$.

Apply the mean value theorem to the function $f(x) = \sin(x)$, on $[\frac{\pi}{6}, \frac{\pi}{5}]$.

Here we want to use an interval that includes $\frac{\pi}{5}$ as one endpoint and the other endpoint being a point where we "know" the value of $\sin(x)$, like $\sin\left(\frac{\pi}{6}\right)$. We could have also used $[\frac{\pi}{5}, \frac{\pi}{4}]$.

As mentioned in the previous example, $f(x) = \sin(x)$ satisfies the conditions of the mean value theorem on this interval.

By the M.V.T. we know there exists a $c$, $\frac{\pi}{6} < c < \frac{\pi}{5}$ such that:




$$\frac{sin\frac{\pi}{5} - sin\frac{\pi}{6}}{\frac{\pi}{5} - \frac{\pi}{6}} = f'(c) = \cos(c)\,; \quad \frac{\pi}{6} < c < \frac{\pi}{5}$$

$$\frac{sin\frac{\pi}{5} - \frac{1}{2}}{\frac{\pi}{30}} = \cos(c) \qquad \frac{\pi}{6} < c < \frac{\pi}{5};$$

since $\cos(x)$ is decreasing on $[\frac{\pi}{6}, \frac{\pi}{4}]$

$$\frac{\sqrt{3}}{2} = cos\frac{\pi}{6} > \cos(c) > cos\frac{\pi}{5} > cos\frac{\pi}{4} = \frac{\sqrt{2}}{2};$$

$$\frac{\sqrt{3}}{2} > \frac{sin\frac{\pi}{5} - \frac{1}{2}}{\frac{\pi}{30}} > \frac{\sqrt{2}}{2}$$

now let's solve the inequality for $sin\frac{\pi}{5}$

$$\frac{\sqrt{3}}{2}\left(\frac{\pi}{30}\right) > \sin\left(\frac{\pi}{5}\right) - \frac{1}{2} > \frac{\sqrt{2}}{2}\left(\frac{\pi}{30}\right)$$

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{30}\right) > \sin\left(\frac{\pi}{5}\right) > \frac{1}{2} + \frac{\sqrt{2}}{2}\left(\frac{\pi}{30}\right)$$

or

$$\frac{1}{2} + \left(\frac{\sqrt{2}}{60}\right)\pi < \sin\left(\frac{\pi}{5}\right) < \frac{1}{2} + \left(\frac{\sqrt{3}}{60}\right)\pi.$$

Ex. Let $f(x) = tan^{-1}x$ and apply the M.V.T. to $[a, b]$; $a, b > 0$ to prove:

a. $$\frac{b-a}{1+b^2} < tan^{-1}b - tan^{-1}a < \frac{b-a}{1+a^2}\ .$$

b. Apply part "a" to $[1, \frac{4}{3}]$ to get the inequality:

$$\frac{\pi}{4} + \frac{3}{25} < tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}.$$

a. $f(x) = tan^{-1}x$ is continuous everywhere and differentiable everywhere and thus it satisfies the M.V. T. on any interval $[a, b]$.

By the M.V.T. there exists a $c$, $0 < a < c < b$ such that:

$$\frac{tan^{-1}b - tan^{-1}a}{b-a} = \frac{1}{1+c^2}; \qquad 0 < a < c < b \quad \text{(since } f'(x) = \frac{1}{1+x^2}\text{)}.$$

Since $0 < a < c < b$ we know that $a^2 < c^2 < b^2$ and
$1 + a^2 < 1 + c^2 < 1 + b^2$,

and finally $\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$ .

Now since $\frac{tan^{-1}b - tan^{-1}a}{b-a} = \frac{1}{1+c^2}$, replacing in the above inequality we get:

$$\frac{1}{1+a^2} > \frac{tan^{-1}b - tan^{-1}a}{b-a} > \frac{1}{1+b^2}\ .$$

Now multiply through by $(b-a)$, which is positive because $b > a$:

$$\frac{b-a}{1+b^2} < tan^{-1}b - tan^{-1}a < \frac{b-a}{1+a^2}\,.$$

b. Applying this inequality when $a = 1$ and $b = \frac{4}{3}$

$$\frac{\frac{4}{3}-1}{1+(\frac{4}{3})^2} < tan^{-1}(\frac{4}{3}) - tan^{-1}1 < \frac{\frac{4}{3}-1}{1+(1)^2}$$

$$\frac{\frac{1}{3}}{\frac{25}{9}} < tan^{-1}(\frac{4}{3}) - \frac{\pi}{4} < \frac{1}{6}$$

$$\frac{\pi}{4} + \frac{3}{25} < tan^{-1}(\frac{4}{3}) < \frac{\pi}{4} + \frac{1}{6}\,.$$

Ex. Prove that $e^x > 1 + x$ for $x > 0$.

Apply the M.V.T. to the function $f(x) = e^x$ on the interval $[0, x]$.

$f(x)$ satisfies the M.V.T. because it's continuous everywhere and differentiable everywhere.

By the M.V.T. we know that there is a $c,\ 0 < c < x$ such that

$$\frac{e^x - e^0}{x - 0} = f'(c) = e^c; \qquad 0 < c < x \quad \text{or}$$

$$\frac{e^x - 1}{x} = e^c; \qquad 0 < c < x.$$

Since $0 < c$ and $f(x) = e^x$ is an increasing function $e^0 = 1 < e^c$. Thus we have:

$\frac{e^x - 1}{x} = e^c > 1$; Now solve this inequality for $e^x$.

$$e^x - 1 > x$$

$e^x > 1 + x$ for $x > 0$.

Ex. Suppose $f'(x)$ exists on $(a, b)$ and $\sup\limits_{a<x<b} |f'(x)| \leq M$, show that $f(x)$ is uniformly continuous on $(a, b)$.

Let $x, y \in (a, b)$ then by the M.V.T. we have:

$$|f(x) - f(y)| \leq M|x - y|.$$

So if we choose $\delta = \epsilon / M$ then:

$$|f(x) - f(y)| \leq M|x - y| < M(\delta) = M\left(\frac{\epsilon}{M}\right) = \epsilon$$

and $f$ is uniformly continuous on $(a, b)$.

Def. A function $f: \mathbb{R} \to \mathbb{R}$ is called **monotonically increasing** if $p > q$ implies that $f(p) \geq f(q)$. $f$ is called **monotonically decreasing** if if $p > q$ implies that $f(p) \leq f(q)$.


Theorem: Suppose $f$ is differentiable in $(a, b)$:

a. If $f'(x) \geq 0$ for all $x\epsilon(a, b)$ then $f$ is monotonically increasing.

b. If $f'(x) = 0$ for all $x\epsilon(a, b)$ then $f$ is a constant function

c. If $f'(x) \leq 0$ for all $x\epsilon(a, b)$ then $f$ is monotonically decreasing.

Proof: Take any two point $p, q\epsilon(a, b)$ with $p > q$. Since $f$ is differentiable in $(a, b)$, it is also differentiable in $(q, p)$ and continuous in $[q, p]$ (if $f$ is differentiable at a point $t$, then it is continuous at $t$) thus $f$ satisfies the conditions of the Mean Value Theorem on $[q, p]$.

Thus we know:

$f(p) - f(q) = f'(c)(p - q)$ where $q < c < p$.

a. If $f'(x) \geq 0$ for all $x\epsilon(a,b)$ then $f'(c) \geq 0$, and thus $f'(c)(p-q) \geq 0$;

Hence $f(p) - f(q) = f'(c)(p-q) \geq 0$ and $f(p) \geq f(q)$.

So $f$ is monotonically increasing.

b. If $f'(x) = 0$ for all $x\epsilon(a,b)$ then $f'(c) = 0$, and thus $f(p) - f(q) = 0$;

or $f(p) = f(q)$. So $f$ is a constant function.

c. If $f'(x) \leq 0$ for all $x\epsilon(a,b)$ then $f'(c) \leq 0$, and thus $f'(c)(p-q) \leq 0$;

Hence $f(p) - f(q) = f'(c)(p-q) \leq 0$ and $f(p) \leq f(q)$.

So $f$ is monotonically decreasing.

Ex. Suppose $f$ is differentiable everywhere and $f(2) = 6$ and $|f'(x)| \leq 4$, for all values of $x$. show that $-6 \leq f(5) \leq 18$ and $-2 \leq f(0) \leq 14$.

Since $f$ is differentiable everywhere it satisfies the Mean Value Theorem on any closed interval $[a,b]$. If we apply the M.V.T. to the interval [2,5] we get:

$$f(5) - f(2) = f'(c)(5-2) \qquad \text{where } 2 < c < 5;$$

Since $f(2) = 6$, we have:

$$f(5) - 6 = (f'(c))(3).$$

Since $|f'(x)| \leq 4$, we know that $-4 \leq f'(x) \leq 4$ for all $x$, so
$-4 \leq f'(c) \leq 4$

and $-12 \leq (f'(c))(3) \leq 12$ .

Since $f(5) - 6 = (f'(c))(3)$ we have:

$$-12 \leq f(5) - 6 \leq 12 \quad \text{or}$$

$$-6 \leq f(5) \leq 18 \text{ .}$$

Now let's apply the M.V.T. to the interval $[0,2]$

$$f(2) - f(0) = f'(c)(2 - 0) \quad \text{where } 0 < c < 2;$$

Since $f(2) = 6$, we have:

$$6 - f(0) = (f'(c))(2).$$

Since $|f'(x)| \leq 4$, we know that $-4 \leq f'(x) \leq 4$ for all $x$, so
$-4 \leq f'(c) \leq 4$

and $-8 \leq (f'(c))(2) \leq 8$ .

Since $6 - f(0) = (f'(c))(2)$ we have:

$$-8 \leq 6 - f(0) \leq 8$$

$$-14 \leq -f(0) \leq 2$$

$$14 \geq f(0) \geq -2 \text{ .}$$

In fact, if $f(x)$ satisfies the Mean Value Theorem on an interval, and $L \leq f'(x) \leq K$ on that interval then we have:

$$L \leq \frac{f(x)-f(a)}{x-a} = f'(c) \leq K.$$

Solving this inequality for $f(x)$ we get:

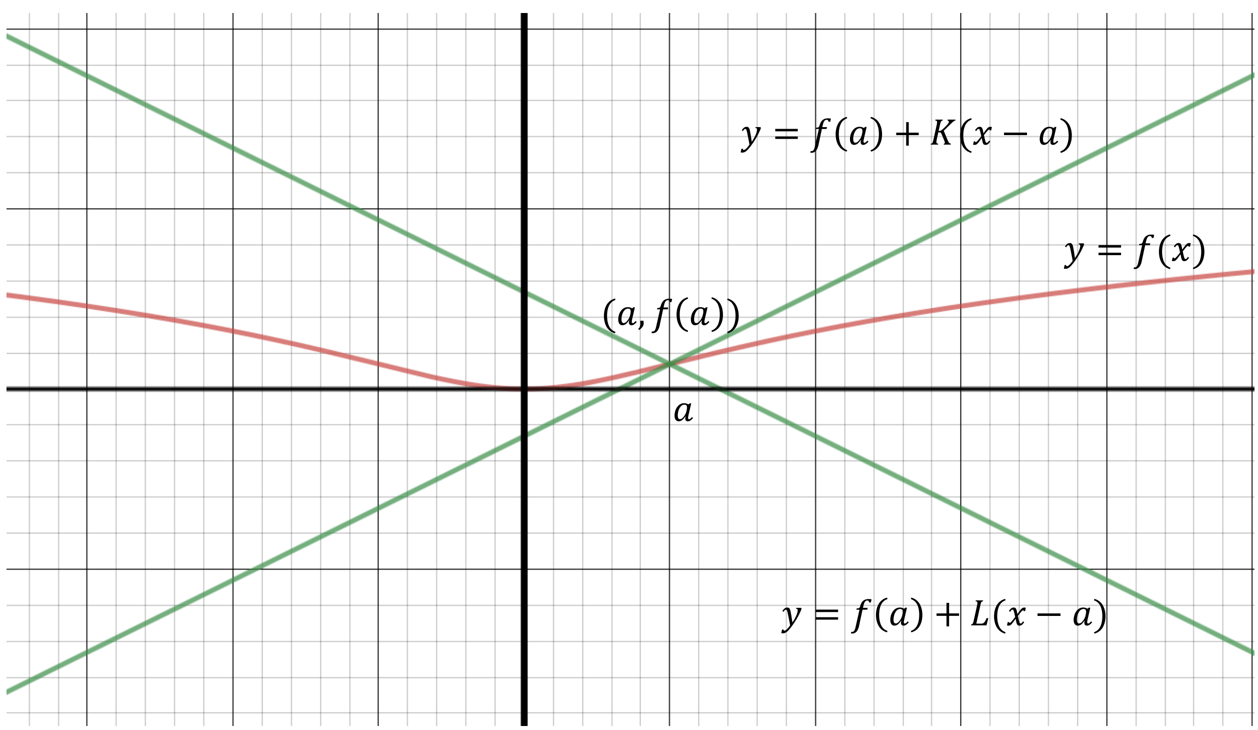
$$f(a) + L(x-a) \leq f(x) \leq f(a) + K(x-a) \qquad \text{if } x \geq a$$

$$f(a) + L(x-a) \geq f(x) \geq f(a) + K(x-a) \qquad \text{if } x \leq a.$$

Which means the values of $f(x)$ can't go outside the lines

$$y = f(a) + L(x-a) \quad \text{and} \quad y = f(a) + K(x-a)$$

on the interval.



Ex. Suppose $f$ is differentiable everywhere and $f(1) = 7$ and $f'(x) \geq -3$, for all values of $x$. Show that $f(6) \geq -8$. Can we find an upper bound on $f(6)$?

Since $f$ is differentiable everywhere it satisfies the Mean Value Theorem on any closed interval $[a, b]$. If we apply the M.V.T. to the interval $[1,6]$ we get:

$f(6) - f(1) = f'(c)(6 - 1)$ where $1 < c < 6$.

Since $f(1) = 7$, we have:

$f(6) - 7 = f'(c)(5)$ where $1 < c < 6$.

Since $f'(x) \geq -3$ for all values of $x$, we know that $f'(c) \geq -3$.

Thus we have:

$f(6) - 7 = f'(c)(5) \geq (-3)(5) = -15$; now add 7 to both sides

$$f(6) \geq -8 .$$

We cannot find an upper bound on $f(6)$ because we have no upper bound on $f'(c)$.

Theorem (L'Hopital's rule) Suppose $f, g$ are real valued differentiable functions on $(a, b)$, and $g'(x) \neq 0$ for all $x \epsilon (a, b)$, where $-\infty \leq a, b \leq +\infty$. Suppose $p \epsilon (a, b)$ (so $p$ could be either $+\infty$ or $-\infty$) and $\lim_{x \to p} \frac{f'(x)}{g'(x)} = A$.

If $\lim_{x \to p} f(x) = 0$, $\lim_{x \to p} g(x) = 0$, or

If $\lim_{x \to p} f(x) = \pm\infty$, $\lim_{x \to p} g(x) = \pm\infty$

Then $\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)} = A$.

Proof: We'll just prove the case where $\lim_{x \to p} f(x) = \lim_{x \to p} g(x) = 0$ and $p \neq \pm\infty$.

Since $\lim_{x \to p} f(x) = \lim_{x \to p} g(x) = 0$ and both $f, g$ are continuous at $x = p$,

$f(p) = g(p) = 0$.

Choose an $x > p$. Since $f, g$ are differentiable everywhere, they satisfy the Extended Mean Value Theore on $[p, x]$, so we can conclude that:

$$\frac{f(x) - f(p)}{g(x) - g(p)} = \frac{f'(c)}{g'(c)} \qquad \text{for } p < c < x.$$

Since $f(p) = g(p) = 0$ we have:

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \qquad \text{for } p < c < x.$$

Now take the limit from the right:

$$\lim_{x\to p^+}\frac{f(x)}{g(x)} = \lim_{x\to p^+}\frac{f'(c)}{g'(c)} = \lim_{x\to p^+}\frac{f'(x)}{g'(x)}.$$

Now Choose an $x < p$. Since $f, g$ are differentiable everywhere, they satisfy the Extended Mean Value Theore on $[x, p]$, so we can conclude that:

$$\frac{f(p)-f(x)}{g(p)-g(x)} = \frac{f'(c)}{g'(c)} \qquad \text{for } x < c < p.$$

Since $f(p) = g(p) = 0$ we have:

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \qquad \text{for } x < c < p.$$

Now take the limit from the left:

$$\lim_{x\to p^-}\frac{f(x)}{g(x)} = \lim_{x\to p^-}\frac{f'(c)}{g'(c)} = \lim_{x\to p^-}\frac{f'(x)}{g'(x)}.$$

Since, by assumption $\lim_{x\to p}\frac{f'(x)}{g'(x)} = A$, the right hand and left hand limits must be the same (for both $\frac{f'(x)}{g'(x)}$ and $\frac{f(x)}{g(x)}$).

Thus we have: $\lim_{x\to p}\frac{f(x)}{g(x)} = \lim_{x\to p}\frac{f'(x)}{g'(x)} = A.$

Ex. Find $\lim_{x\to 2} \frac{x^2-2x}{\sin(x-2)}$ .

$\lim_{x\to 2}(x^2 - 2x) = 0$, and $\lim_{x\to 2} \sin(x - 2) = 0$, so by L'Hopital's rule:

$$\lim_{x\to 2} \frac{x^2-2x}{\sin(x-2)} = \lim_{x\to 2} \frac{2x-2}{\cos(x-2)} = \frac{2}{1} = 2.$$