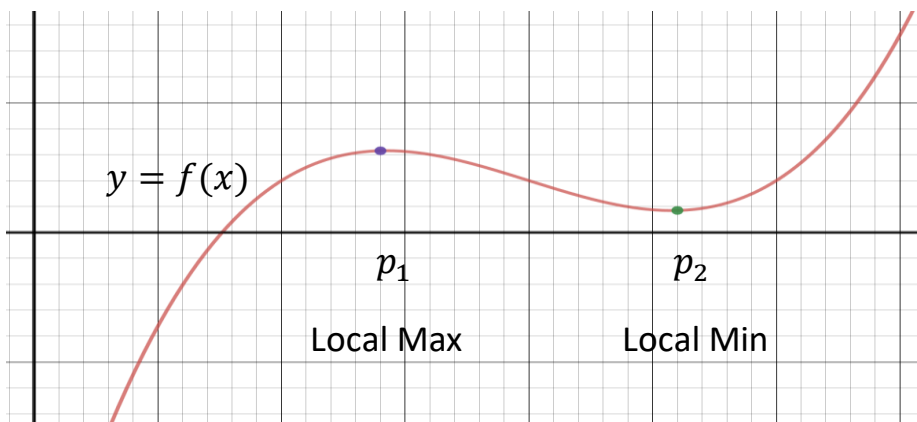


## The Mean Value Theorem

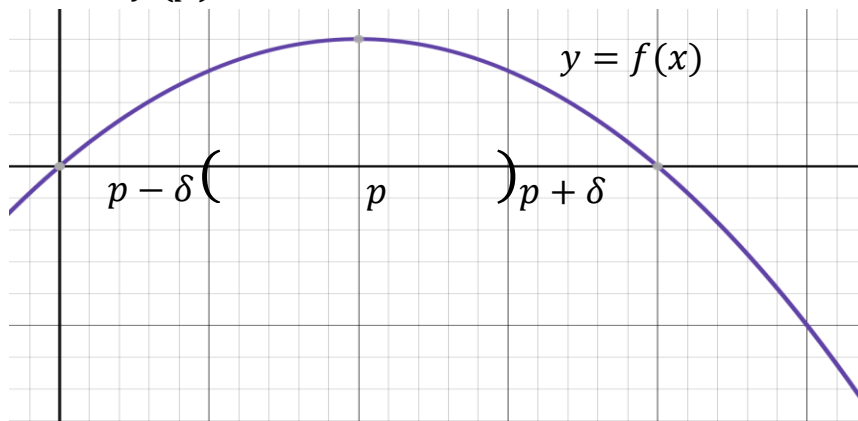
Def. Let  $f$  be a real valued function defined on a metric space  $X$ . We say that  $f$  has a **local maximum** at a point  $p \in X$  if there exists a  $\delta > 0$  such that  $f(q) \leq f(p)$  for all  $q \in X$  with  $d_X(p, q) < \delta$ .

We say that  $f$  has a **local minimum** at a point  $p \in X$  if there exists a  $\delta > 0$  such that  $f(p) \leq f(q)$  for all  $q \in X$  with  $d_X(p, q) < \delta$ .



Theorem: Let  $f: [a, b] \rightarrow \mathbb{R}$ . If  $f$  has a local maximum or minimum at a point  $p \in (a, b)$ , and if  $f'(p)$  exists, then  $f'(p) = 0$ .

Proof: Suppose  $f'(p)$  exists and  $f(p)$  is a local maximum.



Then by the definition of a local maximum, there exists a  $\delta > 0$  such that

$$f(x) \leq f(p) \text{ for all } x \in X \text{ with } |x - p| < \delta.$$

$$|x - p| < \delta$$

$$-\delta < x - p < \delta$$

$$p - \delta < x < p + \delta.$$

Suppose that we take a point  $t$ ,  $p - \delta < t < p$ , then we have:

$$f(t) - f(p) \leq 0 \quad \text{since } f(x) \leq f(p) \text{ for all } x \in X \text{ with } |x - p| < \delta \text{ and}$$

$$t - p < 0 \quad \text{since } t < p;$$

So we have:

$$\frac{f(t) - f(p)}{t - p} \geq 0; \text{ for } t < p. \text{ Thus we can say: } \lim_{t \rightarrow p^-} \frac{f(t) - f(p)}{t - p} \geq 0.$$

Now suppose we take a point,  $p < t < p + \delta$ .

So we have:

$$f(t) - f(p) \leq 0 \quad \text{since } f(x) \leq f(p) \text{ for all } x \in X \text{ with } |x - p| < \delta$$

$$t - p > 0 \quad \text{since } p < t;$$

So we have:

$$\frac{f(t) - f(p)}{t - p} \leq 0; \text{ for } t > p. \text{ That gives us: } \lim_{t \rightarrow p^+} \frac{f(t) - f(p)}{t - p} \leq 0.$$

Since  $f'(p)$  exists we must have:

$$0 \leq \lim_{t \rightarrow p^-} \frac{f(t) - f(p)}{t - p} = \lim_{t \rightarrow p^+} \frac{f(t) - f(p)}{t - p} \leq 0$$

$$\text{Thus } f'(p) = \lim_{t \rightarrow p} \frac{f(t) - f(p)}{t - p} = 0.$$

A similar argument works when  $p$  is a local minimum.

The next theorem will be used later to prove L'Hopital's rule.

Theorem (Generalized Mean Value Theorem): If  $f, g: [a, b] \rightarrow \mathbb{R}$ , are continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there exists a point  $c \in (a, b)$  at which:

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

Note: we could also write this result as:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} ; \text{ This in turn could be written as:}$$

$$\frac{\frac{f(b) - f(a)}{b - a}}{\frac{g(b) - g(a)}{b - a}} = \frac{f'(c)}{g'(c)} \quad \text{i.e.}$$

$$\frac{\text{the average rate of change of } f \text{ over } [a, b]}{\text{the average rate of change of } g \text{ over } [a, b]} = \frac{\text{Inst. rate of change of } f \text{ at } c}{\text{Inst. rate of change of } g \text{ at } c}.$$

Proof: Let  $h(x)$  be defined by:

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x); \quad a \leq x \leq b.$$

$h(x)$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  because  $f(x)$  and  $g(x)$  are.

Notice that  $h(a) = h(b)$ :

$$\begin{aligned} h(a) &= [f(b) - f(a)]g(a) - [g(b) - g(a)]f(a) \\ &= f(b)g(a) - g(b)f(a) \end{aligned}$$

$$\begin{aligned} h(b) &= [f(b) - f(a)]g(b) - [g(b) - g(a)]f(b) \\ &= -g(b)f(a) + f(b)g(a). \end{aligned}$$

If we can find a point  $c \in (a, b)$  where  $h'(c) = 0$  then we would have:

$$\begin{aligned} 0 &= h'(c) = [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) \quad \text{or} \\ [f(b) - f(a)]g'(c) &= [g(b) - g(a)]f'(c) \quad (\text{which is what we are proving}). \end{aligned}$$

So let's show we can find a point  $c \in (a, b)$  where  $h'(c) = 0$ .

If  $h(x)$  is a constant function then  $h'(x) = 0$  for all  $x \in (a, b)$ .

If  $h(x)$  is not a constant function then there is some point  $p$ ,  $a < p < b$  where either  $h(p) > h(a)$  or  $h(p) < h(a)$ .

If  $h(p) > h(a)$ , let  $c$  be a point where  $h$  attains its global maximum (we know a continuous function on a compact set attains its absolute maximum and minimum values),  $a < c < b$ . This global maximum is also a local maximum because it's an interior point. Thus we know from the previous theorem that since  $h(x)$  is differentiable, that  $h'(c) = 0$ .

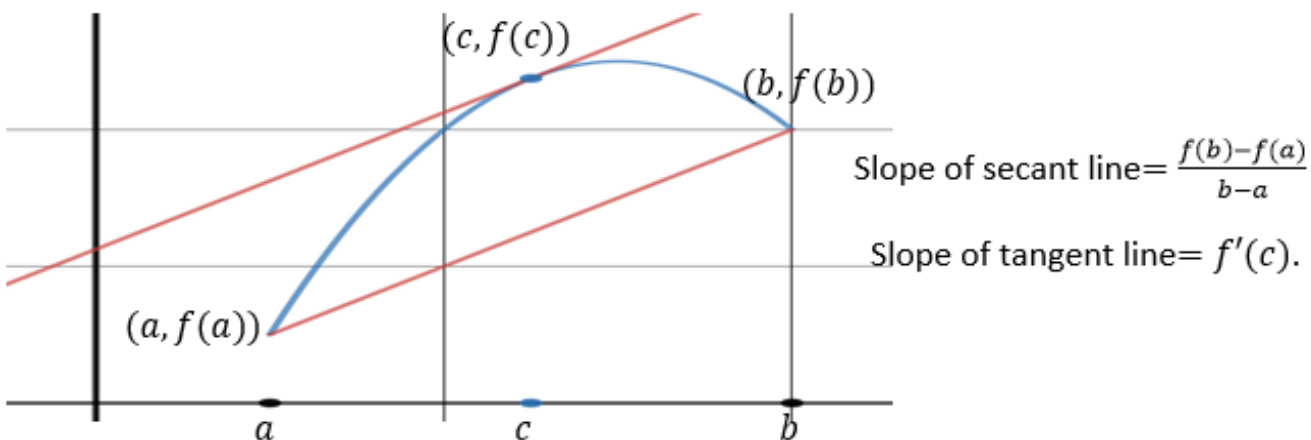
Thus at this point  $c$  we have:

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

A similar argument works for  $h(p) < h(a)$ .

The Mean Value Theorem: Let  $f: [a, b] \rightarrow \mathbb{R}$ , be continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there exists a  $c$ ,  $a < c < b$  such that:

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$



Proof: Let  $g(x) = x$  in the generalized mean value theorem. Then we have:

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

$$[f(b) - f(a)](1) = (b - a)f'(c)$$

$$\frac{f(b)-f(a)}{b-a} = f'(c) ; \quad a < c < b.$$

Note: The Mean Value Theorem gives us a way to bound  $|f(b) - f(a)|$  for a function:

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$f(b) - f(a) = f'(c)(b - a)$$

$$|f(b) - f(a)| = |f'(c)|(b - a)|$$

$$\inf_{a < x < b} |f'(x)|(b - a) \leq |f(b) - f(a)| \leq \sup_{a < x < b} |f'(x)|(b - a).$$

Ex. Prove  $|\sin(b) - \sin(a)| \leq |b - a|$  for all real values of  $a, b$ .

Apply the M.V.T. to  $[a, b]$ , for any  $a, b$ , and the function  $f(x) = \sin(x)$ .

$f(x) = \sin(x)$  is continuous on  $[a, b]$  because it's continuous everywhere. It's differentiable on  $(a, b)$  because it's differentiable everywhere.

By the mean value theorem there exists a  $c$ ,  $a < c < b$  such that:

$$\frac{\sin(b) - \sin(a)}{b - a} = f'(c) = \cos(c) \quad \text{or we can write:}$$

$$\sin(b) - \sin(a) = (\cos(c))(b - a).$$

Now take absolute values:

$$|\sin(b) - \sin(a)| = |\cos(c)||b - a|; \quad \text{now use the fact that } |\cos c| \leq 1$$

$$|\sin(b) - \sin(a)| = |\cos(c)||b - a| \leq |b - a| \quad \text{so we have:}$$

$$|\sin(b) - \sin(a)| \leq |b - a| \quad \text{for all real values of } a, b.$$

Ex. Use the M. V. Theorem to prove that  $\frac{1}{2} + \left(\frac{\sqrt{2}}{60}\right)\pi < \sin\left(\frac{\pi}{5}\right) < \frac{1}{2} + \left(\frac{\sqrt{3}}{60}\right)\pi$ .

Apply the mean value theorem to the function  $f(x) = \sin(x)$ , on  $\left[\frac{\pi}{6}, \frac{\pi}{5}\right]$ .

Here we want to use an interval that includes  $\frac{\pi}{5}$  as one endpoint and the other endpoint being a point where we "know" the value of  $\sin(x)$ , like  $\sin\left(\frac{\pi}{6}\right)$ . We could have also used  $\left[\frac{\pi}{5}, \frac{\pi}{4}\right]$ .

As mentioned in the previous example,  $f(x) = \sin(x)$  satisfies the conditions of the mean value theorem on this interval.

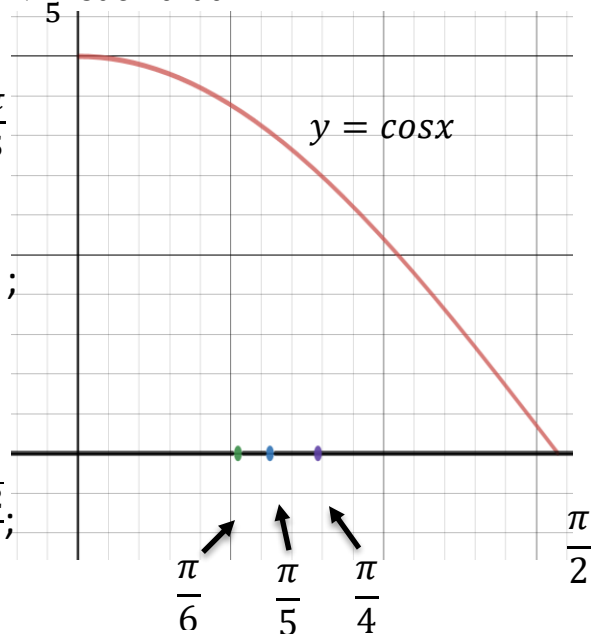
By the M.V.T. we know there exists a  $c$ ,  $\frac{\pi}{6} < c < \frac{\pi}{5}$  such that:

$$\frac{\sin\frac{\pi}{5} - \sin\frac{\pi}{6}}{\frac{\pi}{5} - \frac{\pi}{6}} = f'(c) = \cos(c); \quad \frac{\pi}{6} < c < \frac{\pi}{5}$$

$$\frac{\sin\frac{\pi}{5} - \frac{1}{2}}{\frac{\pi}{30}} = \cos(c) \quad \frac{\pi}{6} < c < \frac{\pi}{5};$$

since  $\cos(x)$  is decreasing on  $[\frac{\pi}{6}, \frac{\pi}{4}]$

$$\frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} > \cos(c) > \cos\frac{\pi}{5} > \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2};$$



$$\frac{\sqrt{3}}{2} > \frac{\sin\frac{\pi}{5} - \frac{1}{2}}{\frac{\pi}{30}} > \frac{\sqrt{2}}{2} \quad \text{now let's solve the inequality for } \sin\frac{\pi}{5}$$

$$\frac{\sqrt{3}}{2} \left(\frac{\pi}{30}\right) > \sin\left(\frac{\pi}{5}\right) - \frac{1}{2} > \frac{\sqrt{2}}{2} \left(\frac{\pi}{30}\right)$$

$$\frac{1}{2} + \frac{\sqrt{3}}{2} \left(\frac{\pi}{30}\right) > \sin\left(\frac{\pi}{5}\right) > \frac{1}{2} + \frac{\sqrt{2}}{2} \left(\frac{\pi}{30}\right) \quad \text{or}$$

$$\frac{1}{2} + \left(\frac{\sqrt{2}}{60}\right)\pi < \sin\left(\frac{\pi}{5}\right) < \frac{1}{2} + \left(\frac{\sqrt{3}}{60}\right)\pi.$$

Ex. Let  $f(x) = \tan^{-1}x$  and apply the M.V.T. to  $[a, b]$ ;  $a, b > 0$  to prove:

a. 
$$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2} .$$

b. Apply part "a" to  $[1, \frac{4}{3}]$  to get the inequality:

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6} .$$

a.  $f(x) = \tan^{-1}x$  is continuous everywhere and differentiable everywhere and thus it satisfies the M.V. T. on any interval  $[a, b]$ .

By the M.V.T. there exists a  $c$ ,  $0 < a < c < b$  such that:

$$\frac{\tan^{-1}b - \tan^{-1}a}{b-a} = \frac{1}{1+c^2}; \quad 0 < a < c < b \quad (\text{since } f'(x) = \frac{1}{1+x^2}).$$

Since  $0 < a < c < b$  we know that  $a^2 < c^2 < b^2$  and  $1 + a^2 < 1 + c^2 < 1 + b^2$ ,

and finally  $\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$ .

Now since  $\frac{\tan^{-1}b - \tan^{-1}a}{b-a} = \frac{1}{1+c^2}$ , replacing in the above inequality we get:

$$\frac{1}{1+a^2} > \frac{\tan^{-1}b - \tan^{-1}a}{b-a} > \frac{1}{1+b^2} .$$



Now multiply through by  $(b - a)$ , which is positive because  $b > a$ :

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}.$$

b. Applying this inequality when  $a = 1$  and  $b = \frac{4}{3}$

$$\frac{\frac{4}{3}-1}{1+(\frac{4}{3})^2} < \tan^{-1}(\frac{4}{3}) - \tan^{-1}1 < \frac{\frac{4}{3}-1}{1+(1)^2}$$

$$\frac{\frac{1}{3}}{\frac{25}{9}} < \tan^{-1}(\frac{4}{3}) - \frac{\pi}{4} < \frac{1}{6}$$

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}(\frac{4}{3}) < \frac{\pi}{4} + \frac{1}{6}.$$

Ex. Prove that  $e^x > 1 + x$  for  $x > 0$ .

Apply the M.V.T. to the function  $f(x) = e^x$  on the interval  $[0, x]$ .

$f(x)$  satisfies the M.V.T. because it's continuous everywhere and differentiable everywhere.

By the M.V.T. we know that there is a  $c$ ,  $0 < c < x$  such that

$$\frac{e^x - e^0}{x - 0} = f'(c) = e^c; \quad 0 < c < x \quad \text{or}$$

$$\frac{e^x - 1}{x} = e^c; \quad 0 < c < x.$$

Since  $0 < c$  and  $f(x) = e^x$  is an increasing function  $e^0 = 1 < e^c$ . Thus we have:

$$\frac{e^x - 1}{x} = e^c > 1; \quad \text{Now solve this inequality for } e^x.$$

$$e^x - 1 > x$$

$$e^x > 1 + x \quad \text{for } x > 0.$$

Ex. Suppose  $f'(x)$  exists on  $(a, b)$  and  $\sup_{a < x < b} |f'(x)| \leq M$ , show that  $f(x)$  is uniformly continuous on  $(a, b)$ .

Let  $x, y \in (a, b)$  then by the M.V.T. we have:

$$|f(x) - f(y)| \leq M|x - y|.$$

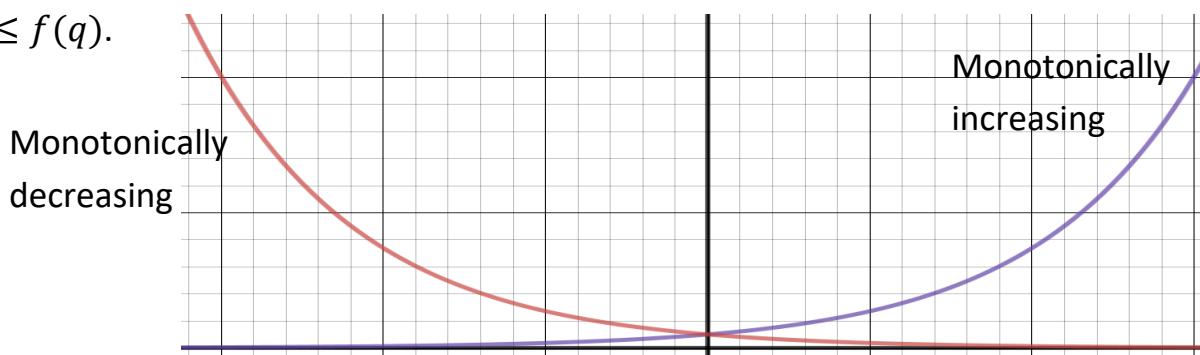
So if we choose  $\delta = \epsilon/M$  then:

$$|f(x) - f(y)| \leq M|x - y| < M(\delta) = M\left(\frac{\epsilon}{M}\right) = \epsilon$$

and  $f$  is uniformly continuous on  $(a, b)$ .

Def. A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is called **monotonically increasing** if  $p > q$  implies that  $f(p) \geq f(q)$ .  $f$  is called **monotonically decreasing** if  $p > q$  implies that

$$f(p) \leq f(q).$$



Theorem: Suppose  $f$  is differentiable in  $(a, b)$ :

- If  $f'(x) \geq 0$  for all  $x \in (a, b)$  then  $f$  is monotonically increasing.
- If  $f'(x) = 0$  for all  $x \in (a, b)$  then  $f$  is a constant function
- If  $f'(x) \leq 0$  for all  $x \in (a, b)$  then  $f$  is monotonically decreasing.

Proof: Take any two point  $p, q \in (a, b)$  with  $p > q$ . Since  $f$  is differentiable in  $(a, b)$ , it is also differentiable in  $(q, p)$  and continuous in  $[q, p]$  (if  $f$  is differentiable at a point  $t$ , then it is continuous at  $t$ ) thus  $f$  satisfies the conditions of the Mean Value Theorem on  $[q, p]$ .

Thus we know:

$$f(p) - f(q) = f'(c)(p - q) \quad \text{where } q < c < p.$$

a. If  $f'(x) \geq 0$  for all  $x \in (a, b)$  then  $f'(c) \geq 0$ , and thus  $f'(c)(p - q) \geq 0$ ;

Hence  $f(p) - f(q) = f'(c)(p - q) \geq 0$  and  $f(p) \geq f(q)$ .

So  $f$  is monotonically increasing.

b. If  $f'(x) = 0$  for all  $x \in (a, b)$  then  $f'(c) = 0$ , and thus  $f(p) - f(q) = 0$ ;

or  $f(p) = f(q)$ . So  $f$  is a constant function.

c. If  $f'(x) \leq 0$  for all  $x \in (a, b)$  then  $f'(c) \leq 0$ , and thus  $f'(c)(p - q) \leq 0$ ;

Hence  $f(p) - f(q) = f'(c)(p - q) \leq 0$  and  $f(p) \leq f(q)$ .

So  $f$  is monotonically decreasing.

Ex. Suppose  $f$  is differentiable everywhere and  $f(2) = 6$  and  $|f'(x)| \leq 4$ , for all values of  $x$ . show that  $-6 \leq f(5) \leq 18$  and  $-2 \leq f(0) \leq 14$ .

Since  $f$  is differentiable everywhere it satisfies the Mean Value Theorem on any closed interval  $[a, b]$ . If we apply the M.V.T. to the interval  $[2, 5]$  we get:

$$f(5) - f(2) = f'(c)(5 - 2) \quad \text{where } 2 < c < 5;$$

Since  $f(2) = 6$ , we have:

$$f(5) - 6 = (f'(c))(3).$$

Since  $|f'(x)| \leq 4$ , we know that  $-4 \leq f'(x) \leq 4$  for all  $x$ , so  
 $-4 \leq f'(c) \leq 4$

and  $-12 \leq (f'(c))(3) \leq 12$ .

Since  $f(5) - 6 = (f'(c))(3)$  we have:

$$-12 \leq f(5) - 6 \leq 12 \quad \text{or}$$

$$-6 \leq f(5) \leq 18.$$

Now let's apply the M.V.T. to the interval  $[0,2]$

$$f(2) - f(0) = f'(c)(2 - 0) \quad \text{where } 0 < c < 2;$$

Since  $f(2) = 6$ , we have:

$$6 - f(0) = (f'(c))(2).$$

Since  $|f'(x)| \leq 4$ , we know that  $-4 \leq f'(x) \leq 4$  for all  $x$ , so

$$-4 \leq f'(c) \leq 4$$

$$\text{and } -8 \leq (f'(c))(2) \leq 8.$$

Since  $6 - f(0) = (f'(c))(2)$  we have:

$$-8 \leq 6 - f(0) \leq 8$$

$$-14 \leq -f(0) \leq 2$$

$$14 \geq f(0) \geq -2.$$

In fact, if  $f(x)$  satisfies the Mean Value Theorem on an interval, and  $L \leq f'(x) \leq K$  on that interval then we have:

$$L \leq \frac{f(x)-f(a)}{x-a} = f'(c) \leq K.$$

Solving this inequality for  $f(x)$  we get:

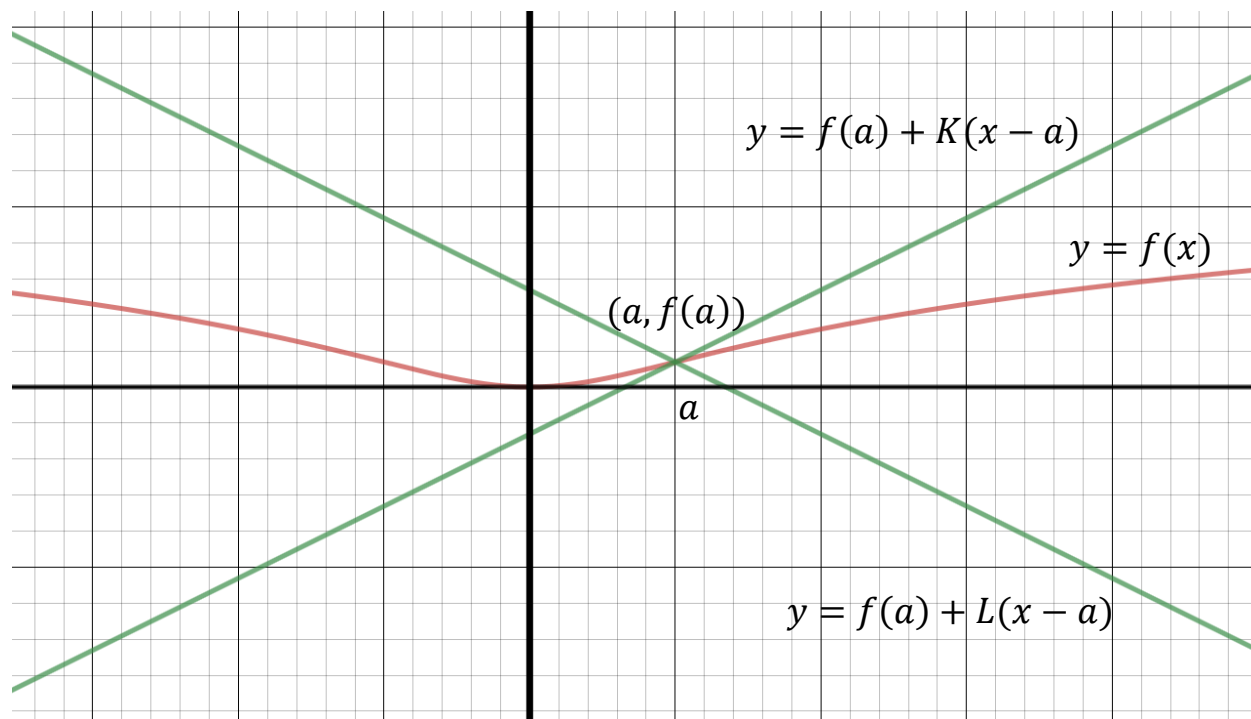
$$f(a) + L(x - a) \leq f(x) \leq f(a) + K(x - a) \quad \text{if } x \geq a$$

$$f(a) + L(x - a) \geq f(x) \geq f(a) + K(x - a) \quad \text{if } x \leq a.$$

Which means the values of  $f(x)$  can't go outside the lines

$$y = f(a) + L(x - a) \quad \text{and} \quad y = f(a) + K(x - a)$$

on the interval.



Ex. Suppose  $f$  is differentiable everywhere and  $f(1) = 7$  and  $f'(x) \geq -3$ , for all values of  $x$ . Show that  $f(6) \geq -8$ . Can we find an upper bound on  $f(6)$ ?

Since  $f$  is differentiable everywhere it satisfies the Mean Value Theorem on any closed interval  $[a, b]$ . If we apply the M.V.T. to the interval  $[1, 6]$  we get:

$$f(6) - f(1) = f'(c)(6 - 1) \quad \text{where } 1 < c < 6.$$

Since  $f(1) = 7$ , we have:

$$f(6) - 7 = f'(c)(5) \quad \text{where } 1 < c < 6.$$

Since  $f'(x) \geq -3$  for all values of  $x$ , we know that  $f'(c) \geq -3$ .

Thus we have:

$$f(6) - 7 = f'(c)(5) \geq (-3)(5) = -15; \quad \text{now add 7 to both sides}$$
$$f(6) \geq -8.$$

We cannot find an upper bound on  $f(6)$  because we have no upper bound on  $f'(c)$ .

Theorem (L'Hopital's rule) Suppose  $f, g$  are real valued differentiable functions on  $(a, b)$ , and  $g'(x) \neq 0$  for all  $x \in (a, b)$ , where  $-\infty \leq a, b \leq +\infty$ . Suppose  $p \in (a, b)$  (so  $p$  could be either  $+\infty$  or  $-\infty$ ) and  $\lim_{x \rightarrow p} \frac{f'(x)}{g'(x)} = A$ .

If  $\lim_{x \rightarrow p} f(x) = 0$ ,  $\lim_{x \rightarrow p} g(x) = 0$ , or

If  $\lim_{x \rightarrow p} f(x) = \pm\infty$ ,  $\lim_{x \rightarrow p} g(x) = \pm\infty$

Then  $\lim_{x \rightarrow p} \frac{f(x)}{g(x)} = \lim_{x \rightarrow p} \frac{f'(x)}{g'(x)} = A$ .

Proof: We'll just prove the case where  $\lim_{x \rightarrow p} f(x) = \lim_{x \rightarrow p} g(x) = 0$  and  $p \neq \pm\infty$ .

Since  $\lim_{x \rightarrow p} f(x) = \lim_{x \rightarrow p} g(x) = 0$  and both  $f, g$  are continuous at  $x = p$ ,

$$f(p) = g(p) = 0.$$

Choose an  $x > p$ . Since  $f, g$  are differentiable everywhere, they satisfy the Extended Mean Value Theorem on  $[p, x]$ , so we can conclude that:

$$\frac{f(x) - f(p)}{g(x) - g(p)} = \frac{f'(c)}{g'(c)} \quad \text{for } p < c < x.$$

Since  $f(p) = g(p) = 0$  we have:

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \quad \text{for } p < c < x.$$



Now take the limit from the right:

$$\lim_{x \rightarrow p^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow p^+} \frac{f'(c)}{g'(c)} = \lim_{x \rightarrow p^+} \frac{f'(x)}{g'(x)}.$$

Now Choose an  $x < p$ . Since  $f, g$  are differentiable everywhere, they satisfy the Extended Mean Value Theorem on  $[x, p]$ , so we can conclude that:

$$\frac{f(p) - f(x)}{g(p) - g(x)} = \frac{f'(c)}{g'(c)} \quad \text{for } x < c < p.$$

Since  $f(p) = g(p) = 0$  we have:

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \quad \text{for } x < c < p.$$

Now take the limit from the left:

$$\lim_{x \rightarrow p^-} \frac{f(x)}{g(x)} = \lim_{x \rightarrow p^-} \frac{f'(c)}{g'(c)} = \lim_{x \rightarrow p^-} \frac{f'(x)}{g'(x)}.$$

Since, by assumption  $\lim_{x \rightarrow p} \frac{f'(x)}{g'(x)} = A$ , the right hand and left hand limits must be the same (for both  $\frac{f'(x)}{g'(x)}$  and  $\frac{f(x)}{g(x)}$ ).

Thus we have:  $\lim_{x \rightarrow p} \frac{f(x)}{g(x)} = \lim_{x \rightarrow p} \frac{f'(x)}{g'(x)} = A$ .

Ex. Find  $\lim_{x \rightarrow 2} \frac{x^2 - 2x}{\sin(x-2)}$ .

$\lim_{x \rightarrow 2} (x^2 - 2x) = 0$ , and  $\lim_{x \rightarrow 2} \sin(x - 2) = 0$ , so by L'Hopital's rule:

$$\lim_{x \rightarrow 2} \frac{x^2 - 2x}{\sin(x-2)} = \lim_{x \rightarrow 2} \frac{2x-2}{\cos(x-2)} = \frac{2}{1} = 2.$$