Continuity and Connectedness

Recall that:

Def. Two subsets A, B of a metric space X, d are said to be **separated** if $A \cap \overline{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$ (i.e., no point of A lies in the closure of B and no point of B lies in the closure of A).

Def. A set $E \subseteq X, d$ a metric space is said to be **connected** if E is not the union of two nonempty separated sets.

Ex. if $A = (0,1)$ and $B = (1,2)$, then A and B are separated sets since
$\bar{A} = [0,1], \bar{B} = [1,2]$
thus:
$$A \cap \bar{B} = (0,1) \cap [1,2] = \emptyset \text{ and}$$
$$\bar{A} \cap B = [0,1] \cap (1,2) = \emptyset.$$
Thus the set $A \cup B = (0,1) \cup (1,2)$ is not a connected set.

Ex. If $A = (0,1]$ and $B = (1,2)$, then A and B are not separated since
$\bar{B} = [1,2]$ and thus

$$A \cap \bar{B} = (0,1] \cap [1,2] = \{1\} \neq \emptyset$$

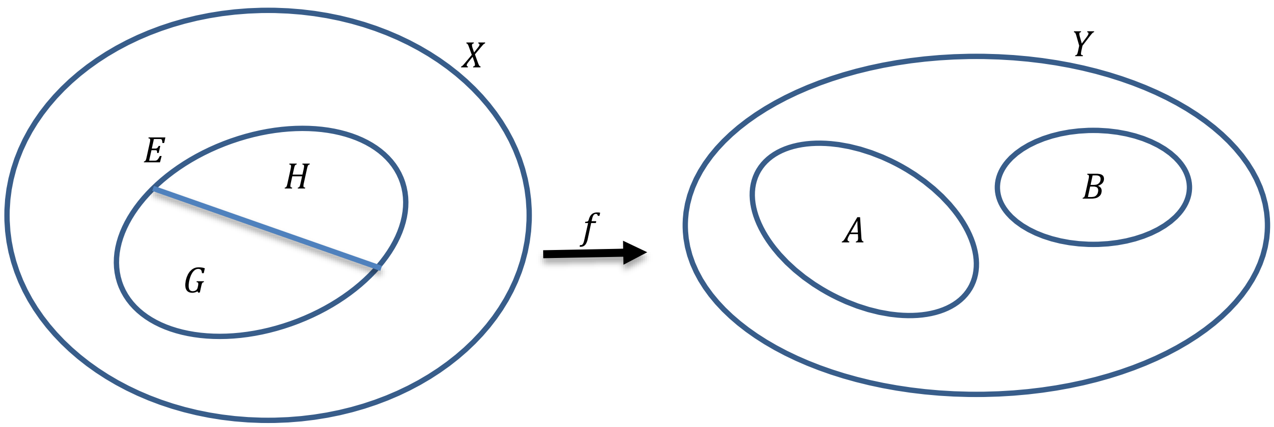
(notice that $\bar{A} \cap B = [0,1] \cap (1,2) = \emptyset$).

Theorem: A subset $E \subseteq \mathbb{R}$ is connected if and only if , if $x \epsilon E$, $y \epsilon E$ and $x < z < y$ then $z \epsilon E$.

Theorem: If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X then $f(E)$ is connected.

Proof: (This will be a proof by contradiction) Assume the contrary, i.e. that f is a continuous mapping and $f(E)$ is not connected.

Thus $f(E) = A \cup B$, where A and B are non-empty separated sets.



Let $G = E \cap f^{-1}(A)$, $\quad H = E \cap f^{-1}(B)$.

Then $E = G \cup H$ and neither G nor H is empty.

Since $A \subseteq \bar{A}$, we have $G \subseteq f^{-1}(\bar{A})$ and $f^{-1}(\bar{A})$ is closed because f is continuous and $\bar{A}$ is closed (inverse image of a closed set is closed when f is continuous).

Since $f^{-1}(\bar{A})$ is closed, $\bar{G} \subseteq f^{-1}(\bar{A})$.

This means that $f(\bar{G}) \subseteq \bar{A}$.

Since $f(H) = B$ and $\bar{A} \cap B = \emptyset$ (A and B are separated sets), $\bar{G} \cap H = \emptyset$.

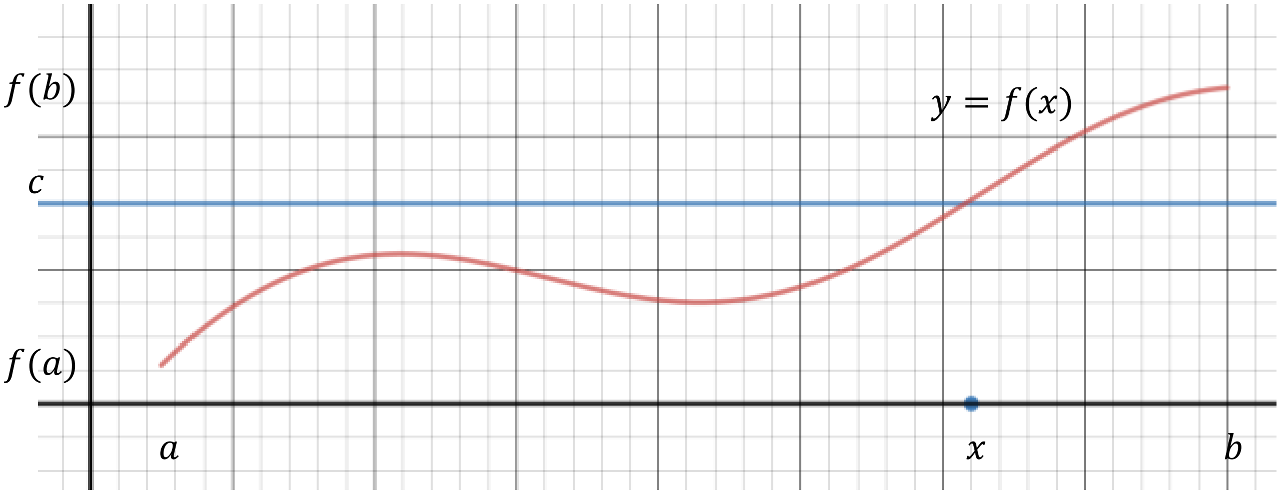
(If $y \epsilon \bar{G} \cap H$, then $f(y) \epsilon \bar{A}$ since $y \epsilon \bar{G}$, and $f(y) \epsilon B$ since $y \epsilon H$, but $\bar{A} \cap B = \emptyset$).

A similar argument shows $G \cap \bar{H} = \emptyset$.

But that would mean that G, H are separated sets with $E = G \cup H$ and thus E is not connected, a contradiction.

Thus $f(E)$ is connected.

Theorem (The Intermediate Value Theorem): Let f be a function $f: \mathbb{R} \to \mathbb{R}$ which is continuous on $[a, b]$. If $f(a) < f(b)$ and if $f(a) < c < f(b)$; then there exists a point $x \epsilon (a, b)$ such that $f(x) = c$.



Proof: $[a, b]$ is connected so $f([a, b])$ is connected because f is continuous on $[a, b]$.

From an earlier theorem we know that for any connected subset E of $\mathbb{R}$, if $x, y \epsilon E$ then for any $x < z < y$, $z \epsilon E$.

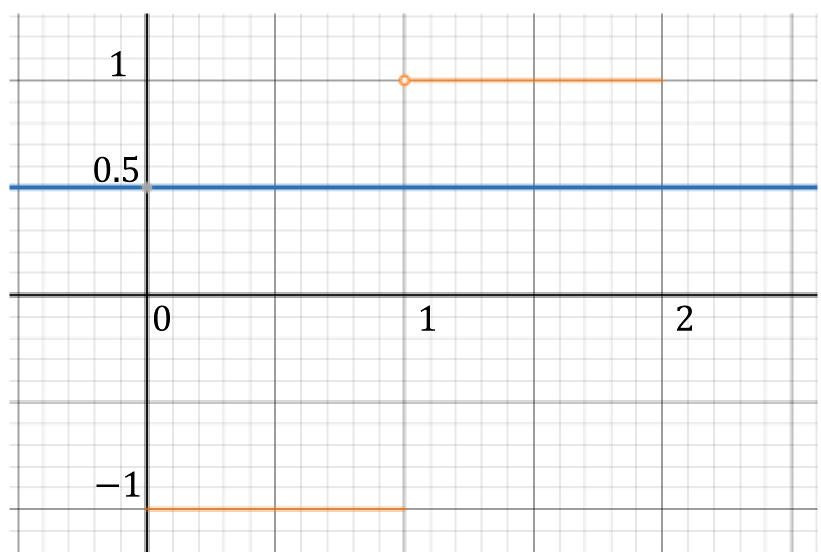
Thus for any $f(a) < c < f(b)$, $c \epsilon f([a, b])$, i.e., there is a point $x \epsilon (a, b)$ such that $f(x) = c$.

Notice that if f is not continuous, there need not be (but there could be) a point $x \epsilon (a, b)$ such that $f(x) = c$.

Ex. $f(x) = 1 \qquad 1 < x \leq 2$

$\qquad = -1 \qquad 0 \leq x \leq 1$

There is no point $x \epsilon (0,2)$ where $f(x) = 0.5$ (or any other value strictly between 1 and -1.

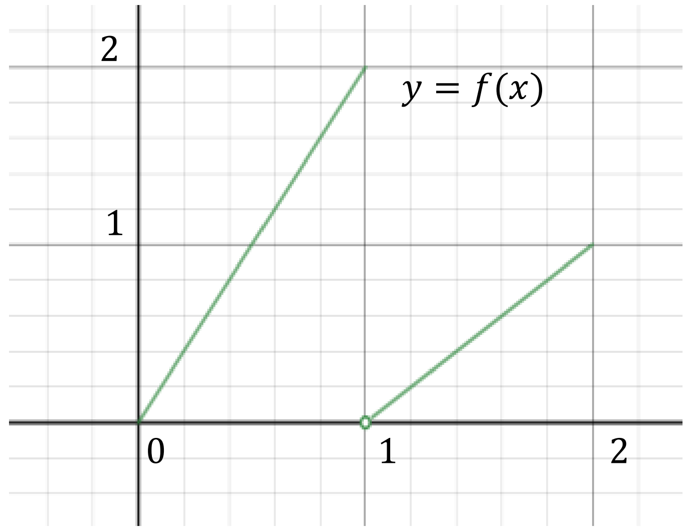


Just because a function is not continuous doesn't mean there can't be a point $x\epsilon(a,b)$ where $f(x)=c$.

Ex. Let $f(x)=x-1 \qquad 1<x\leq 2$

$\qquad\qquad = 2x \qquad 0\leq x\leq 1$

$f(0)=0$ and $f(2)=1$.



Even though $f(x)$ is discontinuous, it's still true that given any $0<c<1$ there is an $x\epsilon(0,2)$ such that $f(x)=c$.

One important applications of the Intermediate Value Theorem is to prove that a continuous function has a root in some interval, i.e. a point where $f(x)=0$.

Ex. Suppose $f(x)=x^8+x^5+x^2-1$. Prove $f(x)$ has a root in $[0,1]$.

$f(x)$ is a polynomial so it is continuous on $[0,1]$ (in fact it's continuous everywhere).

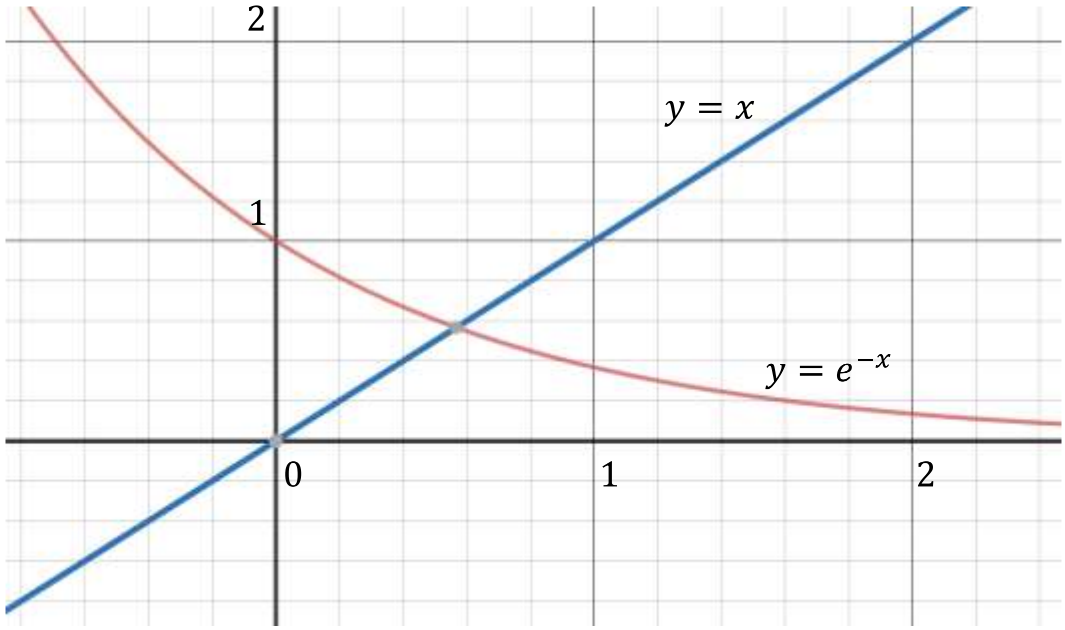
$f(0)=-1$

$f(1)=1+1+1-1=2$.

Since $-1<0<2$, by the intermediate value theorem, there exists an $x\epsilon(0,1)$ such that $f(x)=0$.

Notice that in the previous problem we could also have said there is a point $x\epsilon(0,1)$ such that $f(x)=\frac{\pi}{6}$, since $-1<\frac{\pi}{6}<2$.

Ex. A f function is said to have a "fixed point" if there is some point p where $f(p) = p$. Show that the function $f(x) = e^{-x}$ has a fixed point in the interval $[0,1]$.



This is the same as asking to find that $g(x) = f(x) - x$ has a zero, i.e. that $g(x) = e^{-x} - x$ has a root (or a zero) in $[0,1]$.

$g(x)$ is continuous because e^{-x} and x are continuous functions (we will just accept that e^{-x} is continuous for now).

$g(0) = e^0 - 0 = 1$

$g(1) = e^{-1} - 1 < 0$

Thus by the intermediate value theorem there exists a point $p\epsilon(0,1)$ such that $g(p) = 0$, ie, $e^{-p} - p = 0$, or $e^{-p} = p$.

That is, the function $f(x) = e^{-x}$ has a fixed point in the interval $[0,1]$.