Continuity and Compactness

Def. A mapping $f: E \subseteq X \to \mathbb{R}^k$ is said to be **bounded** if there exists a real number M such that $\|f(x)\| \leq M$ for all $x \epsilon E$.

Ex. $f(x,y) = x^2 + y^2$ is bounded for $E = \{(x,y) |\, |x| < 10,\ |y| \leq 5\}$

since $|f(x,y)| \leq 100 + 25 = 125 = M$;

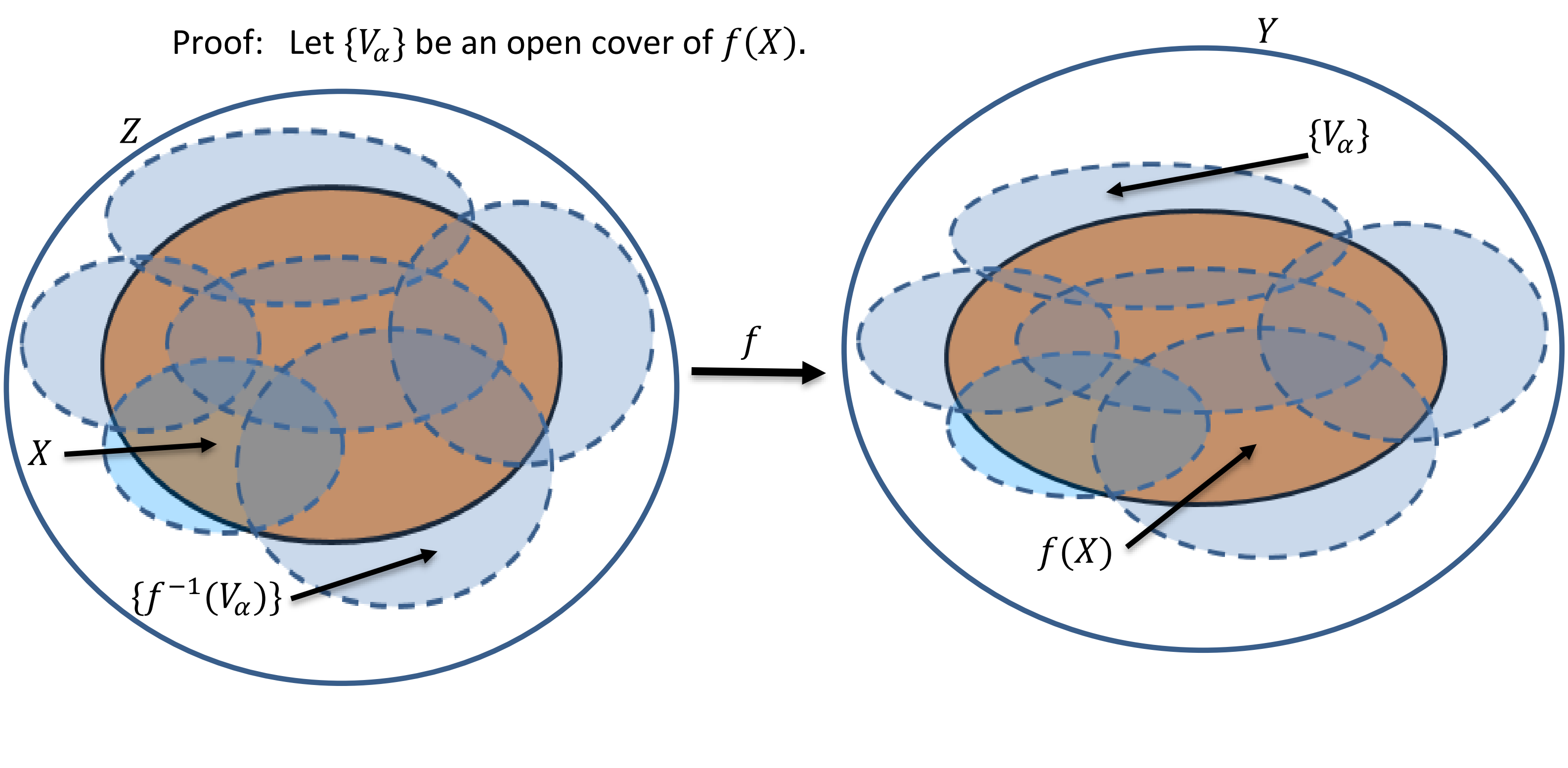
But it is not bounded on $E = \mathbb{R}^2$.

Ex. $f(x,y) = e^{-(x^2+y^2)}$ is bounded for $E = \mathbb{R}^2$ since

$|f(x,y)| = \left|e^{-(x^2+y^2)}\right| \leq 1 = M.$

Theorem: Suppose $f: X \to Y$ is a continuous mapping of a compact metric space X into a metric space Y then $f(X)$ is compact.

Proof: Let $\{V_\alpha\}$ be an open cover of $f(X)$.



Since f is continuous, $f^{-1}(V_\alpha)$, is an open set in X (why?) and
$X \subseteq \bigcup_\alpha f^{-1}(V_\alpha)$.

Thus $\{f^{-1}(V_\alpha)\}$ is an open cover of X.

Since X is compact there exists a finite subcover: $X \subseteq \bigcup_{i=1}^{n} f^{-1}(W_i)$, where

$\{W_i\} \subseteq \{V_\alpha\}$.

Since $f(f^{-1}(E)) \subseteq E$; for $E \subseteq Y$,

(For example, if $f(x) = x^2$ and $E = (-1,1)$; then $f^{-1}(-1,1) = (-1,1)$ and $f(f^{-1}(-1,1)) = [0,1) \subseteq (-1,1)$.)

$f(X) \subseteq \bigcup_{i=1}^{n} f(f^{-1}(W_i)) \subseteq \bigcup_{i=1}^{n} W_i$.

So $\{W_i\}$ is a finite subcover of $f(X)$, and $f(X)$ is compact.

Theorem: Suppose f is a continuous function on a compact metric space X into $\mathbb{R}$, and $M = sup_{p\epsilon X} f(p)$ and $m = inf_{p\epsilon X} f(p)$, then there exist points $p, q\epsilon X$ such that $f(p) = M$ and $f(q) = m$.

Proof: Since f is continuous and X is compact, $f(X)$ is a compact subset of $\mathbb{R}$. By the Heine-Borel theorem we know that any compact subset of $\mathbb{R}$ (or $\mathbb{R}^n$) is closed and bounded.

Hence $f(X)$ contains $M = sup_{p\epsilon X} f(p)$ and $m = inf_{p\epsilon X} f(p)$.

For suppose $M = sup_{p\epsilon X} f(p)$ and $M \notin f(X)$.

Then for every $h > 0$ there exists a point $x \epsilon f(X)$ such that $M - h < x < M$.

Otherwise, $M - h$ would be an upper bound for $f(X)$ and M wouldn't be the least upper bound.

But this means that M is a limit point of $f(X)$. Since $f(X)$ is closed $M \epsilon f(X)$.

A similar argument works for the infimum.

This gives us the theorem from first year Calculus that a continuous function on a closed, bounded interval (i.e. a compact subset of $\mathbb{R}$) takes on its maximum and minimum values.

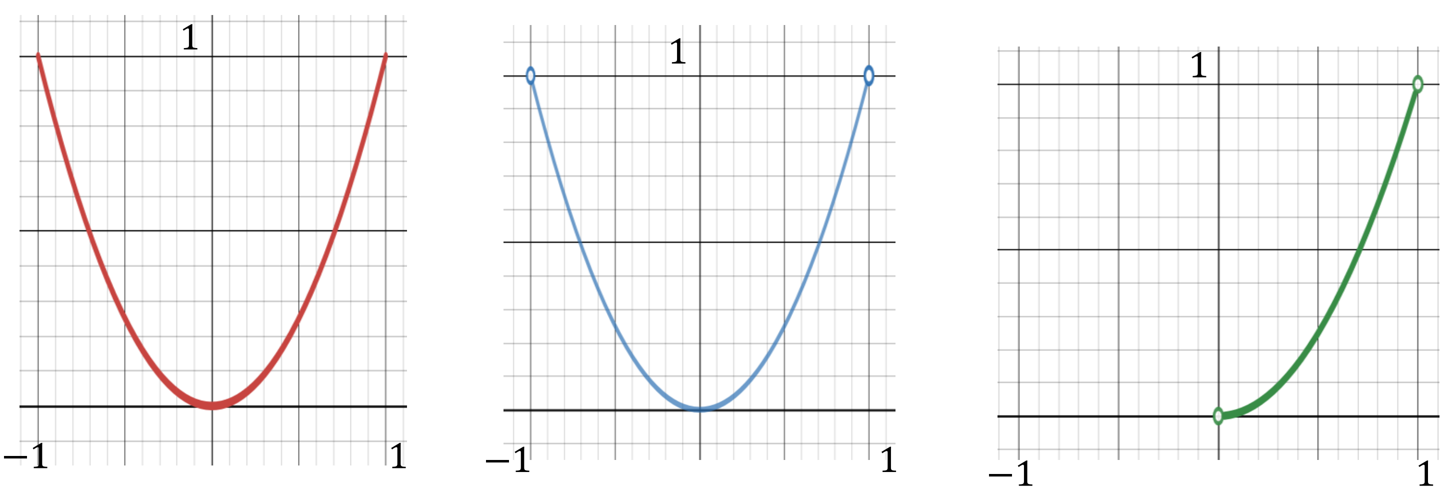
Ex. Let $f(x) = x^2$; and $X = [-1,1]$.

$f(X) = [0,1]$; minimum value=0,   maximum value=1. (In red below)

If $X = (-1,1)$; i.e. X is not compact notice that:

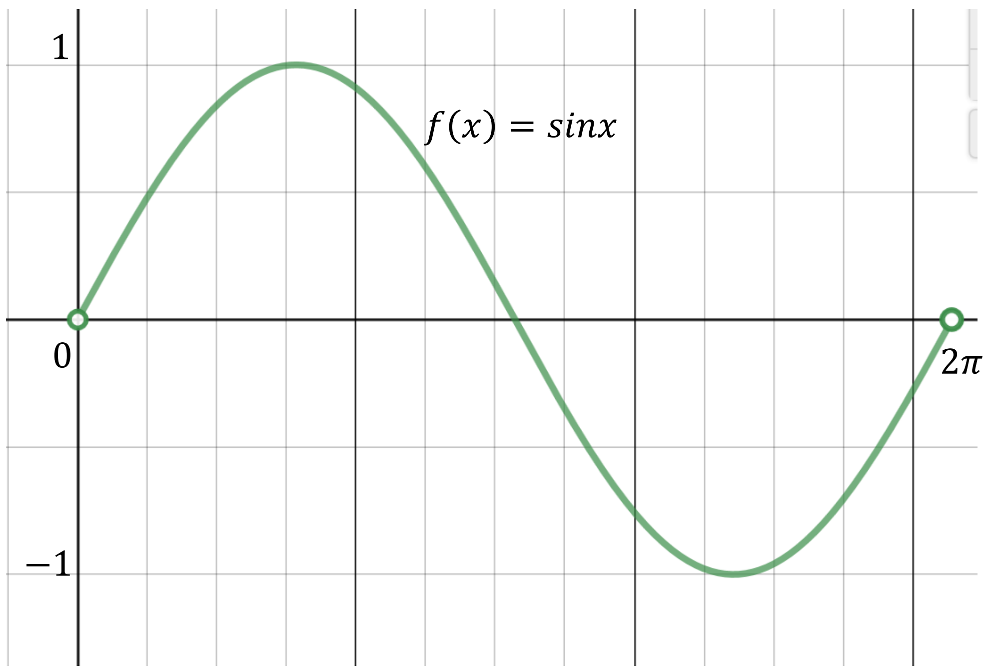
$f(X) = [0,1)$; f takes on its minimum value but not its maximum value. (In blue below)

If $X = (0,1)$, then $f(X) = (0,1)$ and f doesn't take on either its minimum or maximum values. (In green below)



Note: A continuous function on a non-compact set can take on its minimum and/or maximum values, but it does not have to. A continuous function on a compact set must take on its maximum and minimum values.

Ex. Let $f(x) = sinx$; and $X = (0,2\pi)$. Then $f(X) = [-1,1]$. So f takes on its maximum and minimum values even though $X = (0,2\pi)$ is not compact.



Def. Let $f: X \to Y$; X, Y are metric spaces. We say f is **uniformly continuous** on X if for every $\epsilon > 0$ there exists a $\delta > 0$ for all $p, q \in X$ such that if $d_X(p,q) < \delta$ then $d_Y\big(f(p), f(q)\big) < \epsilon$.

For any interval $I \subseteq \mathbb{R}$ with $f: I \subseteq \mathbb{R} \to \mathbb{R}$, f is uniformly continuous on I means for every $\epsilon > 0$ there exists a $\delta > 0$ for all $x, a \in I$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

Notice the difference between continuity and uniform continuity:

1. For uniform continuity, δ does not depend on the point in X you are at. For continuity, the δ can depend on which point in X you are at (with both continuity and uniform continuity, δ does depend on ϵ).

2. Uniform continuity is a property of a set of points, not a single point. Continuity is a property at a point and a set of points.

3. If a function is uniformly continuous on a set X, then it is also continuous on X. However, if a function is continuous on a set X it may, or may not be, uniformly continuous on X.

Ex. Let $f(x) = \frac{1}{x}$; $0 < x < 1$. Show that $f(x)$ is continuous on $(0,1)$ but not uniformly continuous.

To show $f(x) = \frac{1}{x}$ is continuous at any point $a\epsilon(0,1)$ we must show that given any $\epsilon > 0$ we can find a $\delta > 0$ such that if $|x - a| < \delta$ then $\left|\frac{1}{x} - \frac{1}{a}\right| < \epsilon$.

Note that δ can depend on both the value of ϵ and the value of "a".

Let's work backward from the ϵ statement to get the δ statement.

$$\left|\frac{1}{x} - \frac{1}{a}\right| = \left|\frac{a-x}{ax}\right| = \frac{1}{|ax|}|x - a|.$$

We need an upper bound on $\frac{1}{|ax|} = \frac{1}{ax}$; since $a, x > 0$.

Choose $\delta \leq \frac{a}{2}$. (Since $0 < a < 1$, we have to choose a δ neighborhood that stays away from $x = 0$, otherwise $\frac{1}{ax}$ won't be bounded above).

Then we have that $|x - a| < \frac{a}{2}$ or

$$-\frac{a}{2} < x - a < \frac{a}{2} \qquad \text{now add a}$$

$$\frac{a}{2} < x < \frac{3a}{2};$$

Since $\frac{a}{2}, x, \frac{3a}{2} > 0$: $\quad \frac{2}{a} > \frac{1}{x} > \frac{2}{3a}$; $\quad$ now multiply through by $\frac{1}{a} > 0$.

$$\frac{2}{a^2} > \frac{1}{ax} > \frac{2}{3a^2} \implies \frac{1}{|ax|} < \frac{2}{a^2}.$$

Thus we have: if $\delta \leq \frac{a}{2}$ then

$$\left|\frac{1}{x} - \frac{1}{a}\right| = \frac{1}{|ax|}|x-a| < \frac{2}{a^2}|x-a| < \frac{2}{a^2}\delta < \epsilon.$$

$\frac{2}{a^2}\delta < \epsilon$ is equivalent to $\delta < \frac{a^2}{2}\epsilon.$

Choose $\delta = \min\left(\frac{a}{2}, \frac{a^2}{2}\epsilon\right)$ (remember we chose $\delta \leq \frac{a}{2}$ earlier)

Now let's show that this δ works.

If $|x-a| < \delta = \min(\frac{a}{2}, \frac{a^2}{2}\epsilon)$ then we have:

$\left|\frac{1}{x} - \frac{1}{a}\right| = \frac{1}{|ax|}|x-a| < \frac{2}{a^2}|x-a|$ since $\delta \leq \frac{a}{2}$

$\left|\frac{1}{x} - \frac{1}{a}\right| < \frac{2}{a^2}|x-a| < \frac{2}{a^2}\delta \leq \frac{2}{a^2}\left(\frac{a^2}{2}\epsilon\right) = \epsilon$ since $\delta \leq \frac{a^2}{2}\epsilon.$

Thus we have shown that $f(x) = \frac{1}{x}$ continuous at $a\epsilon(0,1)$.

Now let's show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$.

Let's fix an $\epsilon > 0$.

To be uniformly continuous we need to find a $\delta > 0$, that depends only on ϵ, such that if $|x-a| < \delta$ then $\left|\frac{1}{x} - \frac{1}{a}\right| < \epsilon$ for all $a, x \in (0,1)$.

But if $\epsilon > 0$ is fixed, regardless of what δ one chooses, by moving "a" toward 0

$$\left|\frac{1}{x} - \frac{1}{a}\right| = \frac{1}{|ax|}|x - a| \to \infty \text{ for } |x - a| < \delta.$$

So δ must depend on "a" and $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$.

Ex. Show $f(x) = x^2$ is uniformly continuous on $[-1,1]$.

We must show that given any $\epsilon > 0$ there exists a $\delta > 0$ for all $a, x \in [-1,1]$ such that if $|x - a| < \delta$ then $|x^2 - a^2| < \epsilon$.

Let's start with the ϵ statement:

$|x^2 - a^2| = |x - a||x + a|$.

But we also know that $a, x \in [-1,1]$, so $|a| \leq 1$ and $|x| \leq 1$.

Now using the triangle inequality: $|x + a| \leq |x| + |a| \leq 1 + 1 = 2$.

So $|x^2 - a^2| = |x - a||x + a| \leq 2|x - a| < 2\delta$.

So if we can force $2\delta < \epsilon$, we'll almost be done.

So if we choose $\delta < \frac{\epsilon}{2}$ (notice δ doesn't depend on a) we have:

$|x^2 - a^2| = |x - a||x + a| \leq 2|x - a|$ because $|a| \leq 1$ and $|x| \leq 1$

$|x^2 - a^2| \leq 2|x - a| < 2\delta < 2\left(\frac{\epsilon}{2}\right) = \epsilon$ because $< \frac{\epsilon}{2}$.

Hence $f(x) = x^2$ is uniformly continuous on $[-1,1]$.

Theorem: Let $f: X \to Y$, be continuous, X, Y metric spaces with X compact, then f is uniformly continuous on X.

Notice that $f(x) = x^2$ is uniformly continuous on $(-1,1)$ (as well as on $[-1,1]$) even though $(-1,1)$ is not compact. The same δ, ϵ argument that shows that $f(x) = x^2$ is uniformly continuous on $[-1,1]$ also show that it's uniformly continuous on $(-1,1)$. Thus a continuous function on a compact set **must** be uniformly continuous on the compact set. A continuous function on a non-compact set, may or may not be uniformly continuous on that set.

Some special properties of uniformly continuous functions:

1. If $f: E \to Y$ is uniformly continuous on a metric space E and $\{p_n\}$ is a Cauchy sequence in E, then $\{f(p_n)\}$ is a Cauchy sequence in Y.
   Notice that $f(x) = \frac{1}{x}$ is continuous on $(0, \infty)$, but not uniformly continuous. $\{\frac{1}{n}\}$ is a Cauchy sequence in $(0, \infty)$, but $\left\{f\left(\frac{1}{n}\right)\right\} = \{n\}$ is not.
2. If $f: E \subseteq \mathbb{R} \to \mathbb{R}$ is uniformly continuous, E a bounded interval, then $\int_E f(x)dx$ is finite.
   $f(x) = \frac{1}{x}$ is continuous on $(0,1)$, but not uniformly continuous. $\int_0^1 \frac{1}{x} dx$ is not finite.

Notice also that you can have bounded continuous functions that are not uniformly continuous (e.g. $f(x) = \sin\left(\frac{1}{x}\right); \quad 0 < x < 2\pi$) and unbounded continuous functions that are uniformly continuous (e.g. $f(x) = x; \ -\infty < x < \infty$).

Ex. Prove that $f\ (x) = \frac{x}{1-x}$ is uniformly continuous on $[2, \infty)$.

We must show given any $\epsilon > 0$ there exists a $\delta > 0$ for all $a, x \in [2, \infty)$ such that if $|x - a| < \delta$ then $\left|\frac{x}{1-x} - \frac{a}{1-a}\right| < \epsilon$.

Let's start with the ϵ statement:

$$\left|\frac{x}{1-x} - \frac{a}{1-a}\right| = \left|\frac{x(1-a)-a(1-x)}{(1-x)(1-a)}\right| = \left|\frac{(x-a)}{(1-x)(1-a)}\right|$$

$$= |x - a|\left|\frac{1}{(1-x)(1-a)}\right|.$$

Now we must find an upper bound on $\left|\frac{1}{(1-x)(1-a)}\right|$ independent of "a".

Since $2 \le x$ and $2 \le a$: $\quad -1 \le \frac{1}{1-x} < 0$ and $-1 \le \frac{1}{1-a} < 0$.

Thus we can say: $\quad 0 < (\frac{1}{1-x})(\frac{1}{1-a}) \le 1.$

Thus we have:

$$\left|\frac{x}{1-x} - \frac{a}{1-a}\right| = |x - a|\left|\frac{1}{(1-x)(1-a)}\right| \le 1|x - a| = \delta.$$

So if we can force $\delta < \epsilon$ we'll almost be done.

So choose $\delta < \epsilon$ which is independent of "a".

Now let's show $\delta < \epsilon$ works.

If $|x - a| < \delta < \epsilon$ then:

$\left|\frac{x}{1-x} - \frac{a}{1-a}\right| = |x - a| \left|\frac{1}{(1-x)(1-a)}\right| < |x - a|$ because $2 \le x$ and $2 \le a$

$\left|\frac{x}{1-x} - \frac{a}{1-a}\right| < |x - a| < \delta < \epsilon$ because $\delta < \epsilon$.

Hence $f(x) = \frac{x}{1-x}$ is uniformly continuous on $[2, \infty)$.