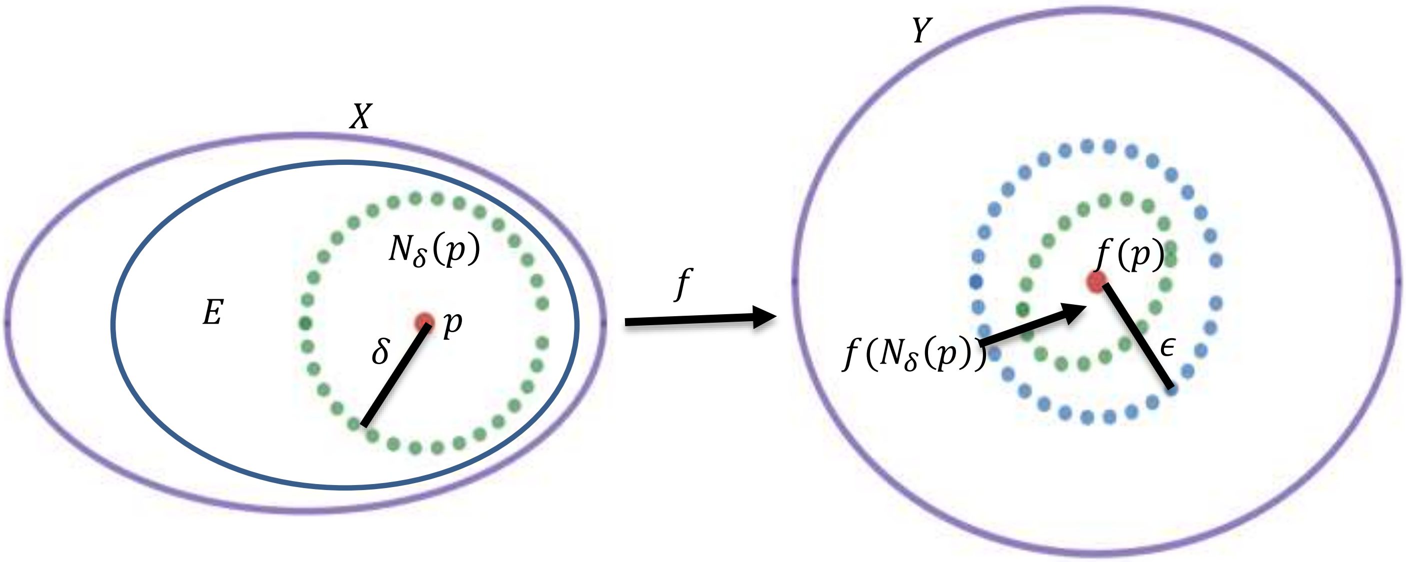
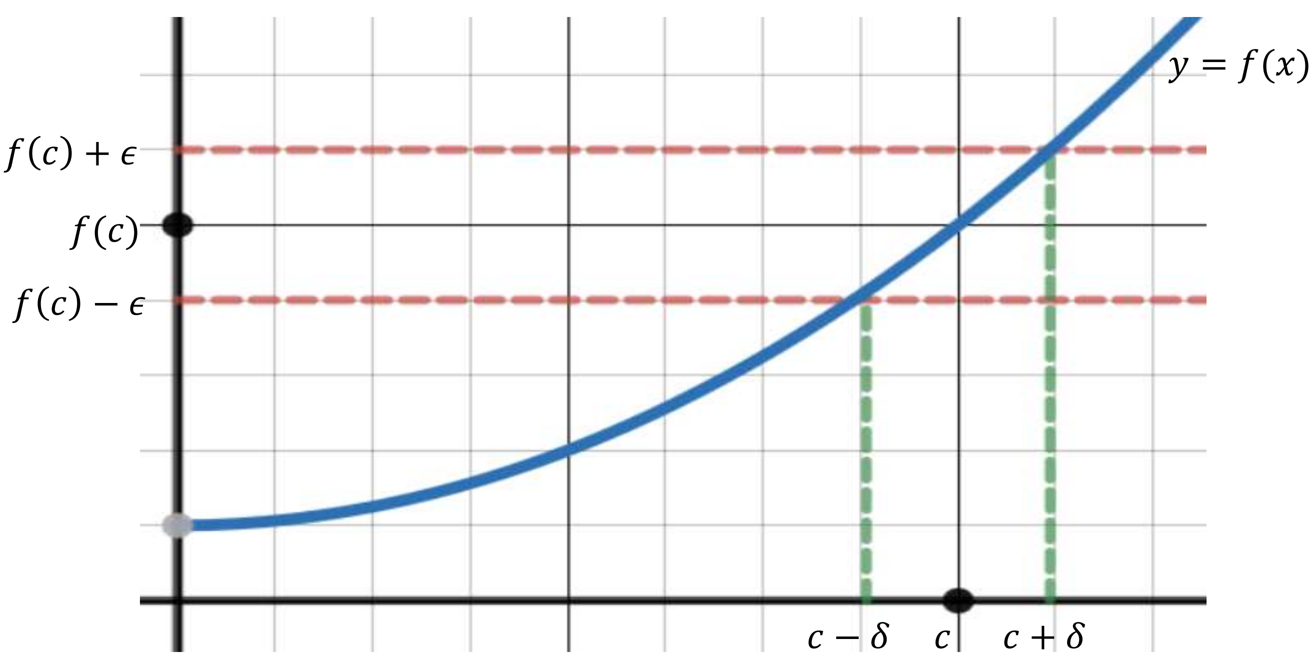
# Continuity

Def. Suppose $X$ and $Y$ are metric spaces, $E \subseteq X$, $p \epsilon E$, and $f: E \to Y$. Then $f$ is said to be **Continuous at $\boldsymbol{p}$** if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all points $x \epsilon E$, if $d_X(x, p) < \delta$ then $d_Y(f(x), f(p)) < \epsilon$. Equivalently, we can say that $f$ is **Continuous at $\boldsymbol{p}$** if $\lim_{x \to p} f(x) = f(p)$.




If $X = Y = \mathbb{R}$ then $f(x)$ is **Continuous at $\boldsymbol{x = c}$** means for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.



Def. If $f$ is Continuous at every point of $E$, then $f$ is said to be **Continuous on $\boldsymbol{E}$**.

Note: For $\lim_{x\to p} f(x)$ to exist, $f(p)$ does not need to be defined (although it can be). For $f(x)$ to be continuous at $p\epsilon E$, $f(p)$ must be defined and equal to $\lim_{x\to p} f(x)$.

If $p\epsilon E$ is an isolated point (i.e., there exists a neighborhood of $p, N(p) \subseteq X$, such that $N(p) \cap E = \{p\}$) then every function that has $p$ in its domain is continuous at $p\epsilon E$. We can see this by choosing $\delta > 0$ such that $d_X(x,p) < \delta$ implies $x = p$.

Then $d_Y(f(x), f(p)) = 0 < \epsilon$.

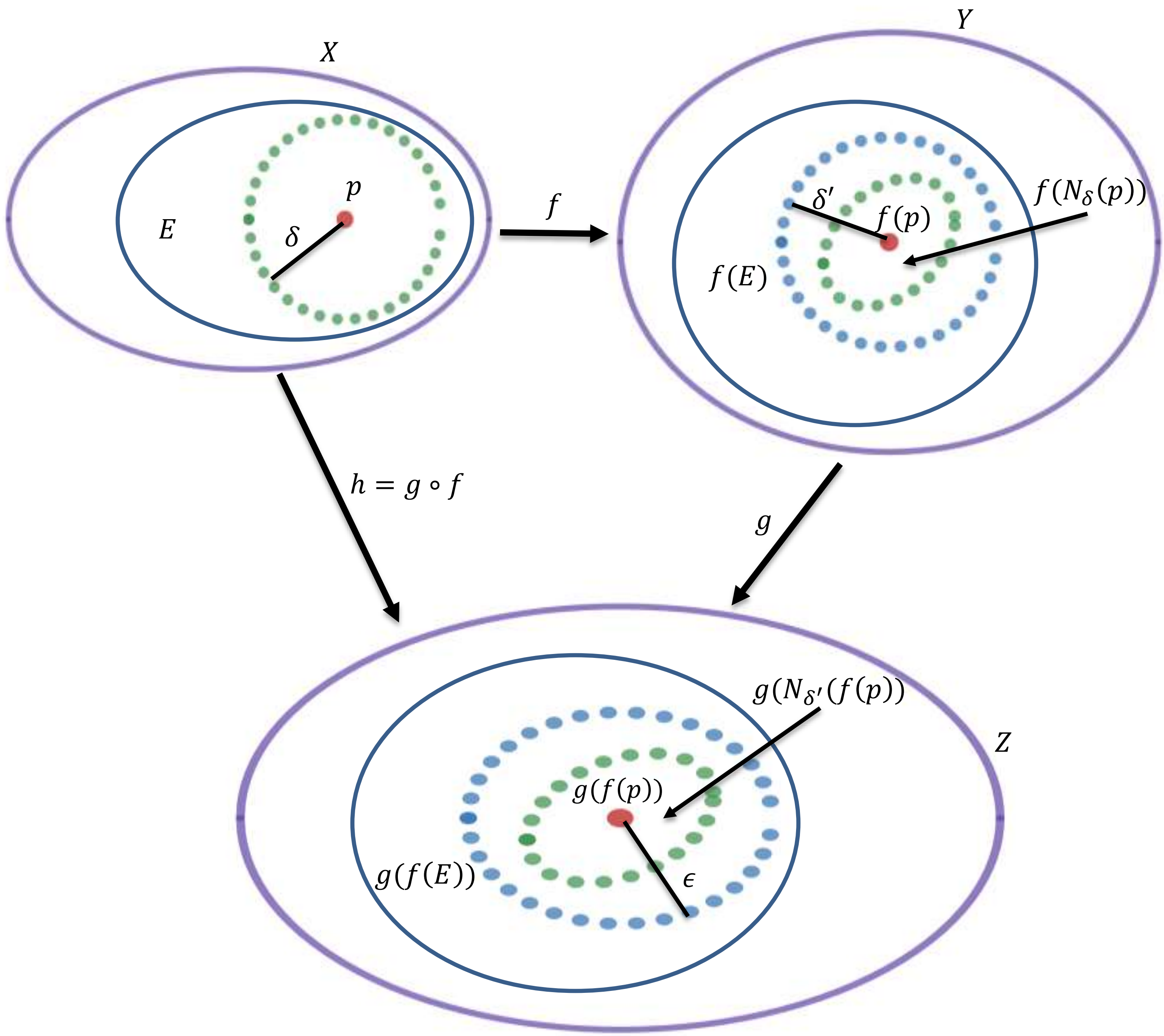
Ex. Let $E = [-1,1] \cup \{5\} \subseteq \mathbb{R}$; and $f: E \to \mathbb{R}$ is any function. Show that $f$ is continuous at $x = 5$.

Given any $\epsilon > 0$, if $\delta < 3$, for example, and $x \in E$, then $d(x,5) < 3$ implies that $x = 5$, and thus $|f(x) - f(5)| = |f(5) - f(5)| = 0 < \epsilon$.

Thus $f$ is continuous at $x = 5$.

Theorem: Suppose $X, Y, Z$ are metric spaces with $E \subseteq X$, $f: E \to Y$ and $g: f(E) \to Z$. Let $h: E \to Z$ by $h(x) = g(f(x))$ for $x \epsilon E$. If $f$ is continuous at $p \epsilon E$ and if $g$ is continuous at $f(p) \epsilon Y$, then $h$ is continuous at $p \epsilon E$.

Proof: We must show that given any $\epsilon > 0$ there exists a $\delta > 0$ such that for all points $x \epsilon E$, if $d_X(x, p) < \delta$ then $d_Z(h(x), h(p)) = d_Z(g(f(x)), g(f(p))) < \epsilon$.



Since $g$ is continuous at $f(p)$, we know we can find a $\delta' > 0$ such that if $d_Y(y, f(p)) < \delta'$ then $d_Z(g(y), g(f(p))) < \epsilon$.

Since $f$ is continuous at $p\epsilon E$, we know we can find a $\delta > 0$ such that if

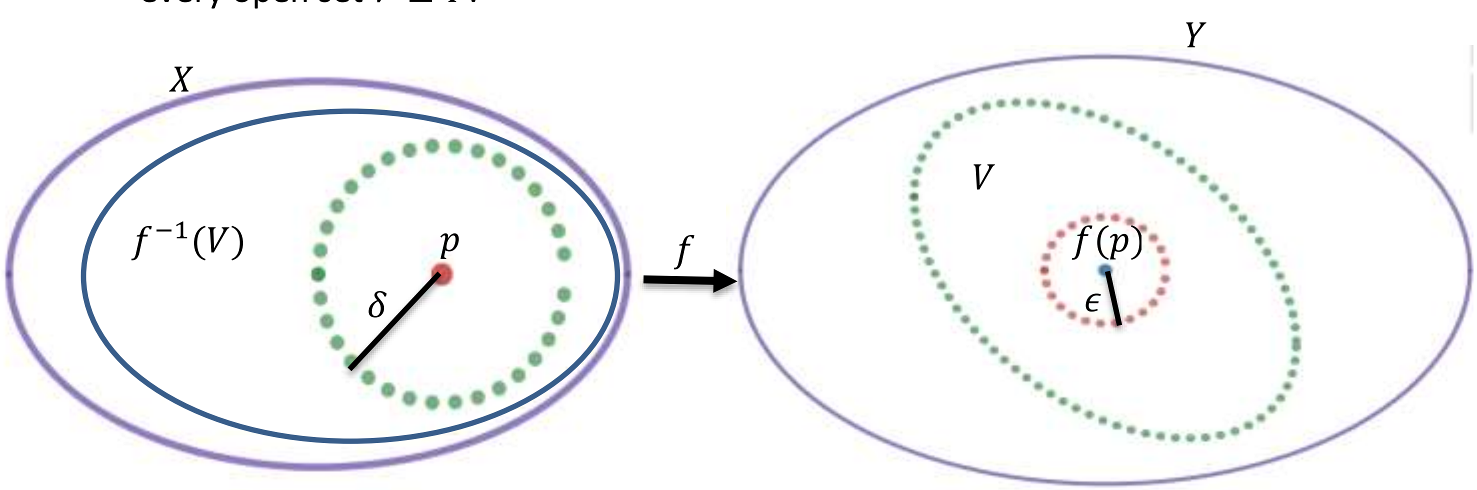
$d_X(x, p) < \delta$ then $d_Y(f(x), f(p)) < \delta'$ for all $x\epsilon E$.

But this means that if $d_X(x, p) < \delta$ then $d_Y(f(x), f(p)) < \delta'$ for all $x\epsilon E$, which in turn means that $d_Z(g(f(x)), g(f(p))) < \epsilon$.

Hence we have shown that $h(x) = g(f(x))$ is continuous at $x = p$.

Theorem: A mapping $f: X \to Y$, $X, Y$ metric spaces is continuous if and only if $f^{-1}(V)$ is open in $X$ for every open set $V \subseteq Y$.

Proof: First assume $f$ is continuous on $X$ and show that $f^{-1}(V)$ is open in $X$ for every open set $V \subseteq Y$.



Let $V$ be any open subset of $Y$. We have to show that every point $p$ in $f^{-1}(V)$ is an interior point of $f^{-1}(V)$.

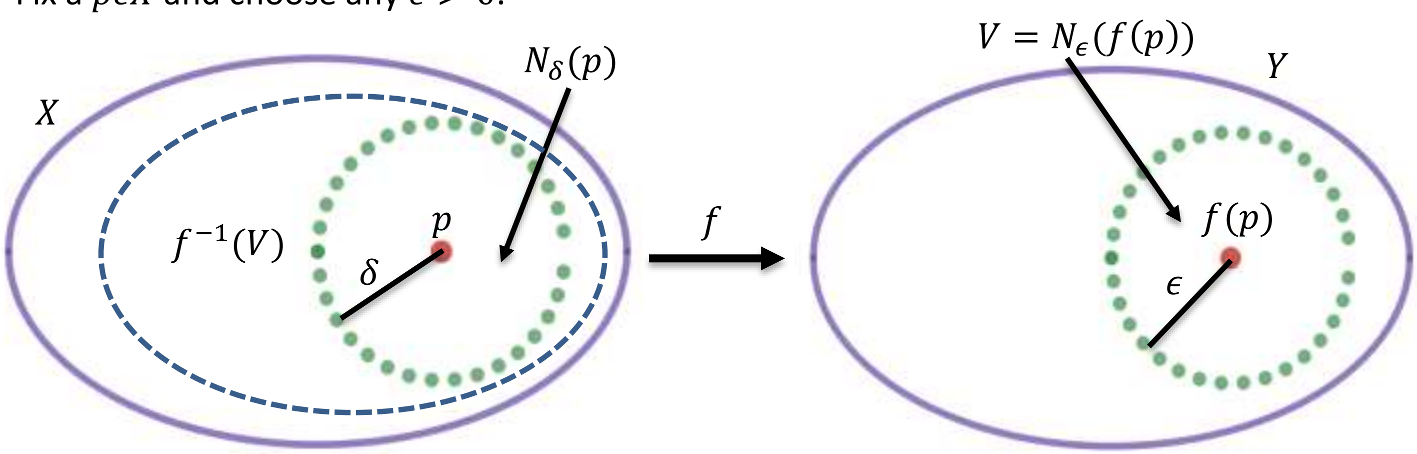
Suppose $p\epsilon f^{-1}(V)$. Since $V$ is open, there exists an $\epsilon > 0$ such that if $d_Y(f(p), y) < \epsilon$ then $y\epsilon V$ (this just says that since $V$ is open, we can find a neighborhood of $f(p)$ that lies entirely inside $V$).

Since $f$ is continuous at $p$, there exists a $\delta > 0$ such that if $d_X(x,p) < \delta$ then $d_Y\big(f(x), f(p)\big) < \epsilon$ , thus $x\epsilon f^{-1}(V)$.

Thus $p$ is an interior point of $f^{-1}(V)$, and $f^{-1}(V)$ is open.

Now let's assume that $f^{-1}(V)$ is open in $X$ for every open set $V \subseteq Y$ and prove that $f$ is a continuous function on $X$.

Fix a $p\epsilon X$ and choose any $\epsilon > 0$.




We need to show that we can find a $\delta > 0$ such that if $d_X(x,p) < \delta$ then $d_Y(f(x), f(p)) < \epsilon$.

Let $V$ be the set of all $y\epsilon Y$ such that $d_Y(y, f(p)) < \epsilon$.

$V$ is an open set in $Y$ (since it's a neighborhood of a point) and hence, by assumption, $f^{-1}(V)$ is open in $X$.

Since $f^{-1}(V)$ is open there exists a $\delta > 0$ such that if $d_X(x,p) < \delta$ then $x \in f^{-1}(V)$.

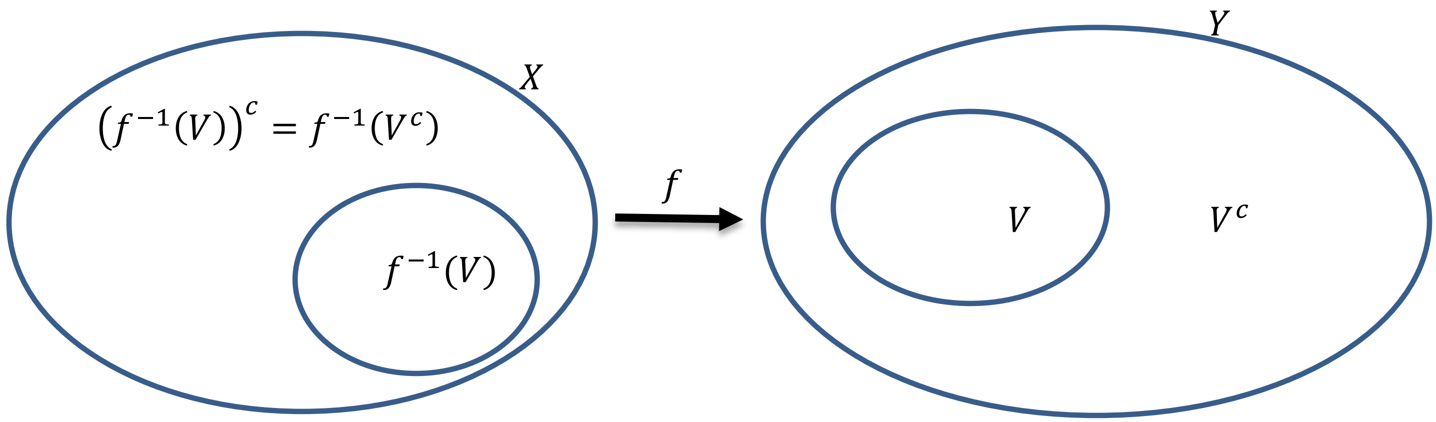
But if $x \epsilon f^{-1}(V)$ then $f(x) \epsilon V$ which means that $d_Y(f(x), f(p)) < \epsilon$.

Hence $f$ is continuous at $p \epsilon X$ for every p.

Thus $f$ is continuous on $X$.

Cor. A mapping $f: X \to Y$, $X, Y$ metric spaces is continuous if and only if $f^{-1}(V)$ is closed in $X$ for every closed set $V \subseteq Y$.

Proof: If $V$ is closed then $V^c$ is open. Thus by the theorem $f$ is continuous if and only if $f^{-1}(V^c)$ is open. The corollary follows from the fact that $f^{-1}(V^c) = \left(f^{-1}(V)\right)^c$.




Note: If $f: X \to Y$ is continuous on $X$, it does NOT imply that:

1. if $V \subseteq X$ is open then $f(V) \subseteq Y$ is open
2. If $W \subseteq X$ is closed then $f(W) \subseteq Y$ is closed.

Ex. Let $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ is continuous at every point (we will show this shortly) in $\mathbb{R}$. However, if $V = (-2, 2)$, which is open in $\mathbb{R}$, then $f(V) = [0,4)$ which is not open in $\mathbb{R}$.

Ex. Let $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{1}{1+x^2}$ is continuous at every point in $\mathbb{R}$. However, if $W = [0, \infty)$, which is closed in $\mathbb{R}$, then $f(W) = (0,1]$ which is not closed in $\mathbb{R}$.

Ex. Prove that $f(x) = x^2$ is continuous at $x = 0$ and $x = a$.

To prove that $f(x) = x^2$ is continuous at $x = 0$ we must show that given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - 0| < \delta$ then $|x^2 - 0| < \epsilon$, i.e. we must prove that $\lim_{x \to 0} x^2 = 0$.

Let's start with the $\epsilon$ statement and work backwards to the $\delta$ statement.

$$|x^2 - 0| = |x|^2 < \epsilon \quad \text{or} \quad |x| < \sqrt{\epsilon}.$$

Now choose $\delta = \sqrt{\epsilon}$.

Now let's show that this $\delta$ works.

If $|x - 0| = |x| < \delta = \sqrt{\epsilon}$ then

$|x^2 - 0| = |x|^2 < \epsilon$

Hence $\lim_{x \to 0} x^2 = 0$, and $f(x) = x^2$ is continuous at $x = 0$.

To prove that $f(x) = x^2$ is continuous at $x = a$ we must show that given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta$ then $|x^2 - a^2| < \epsilon$, i.e. we must prove that $\lim_{x \to a} x^2 = a^2$.

Let's start with the $\epsilon$ statement and work backwards the $\delta$ statement.

$$|x^2 - a^2| = |(x + a)(x - a)| = |x + a||x - a|$$

$|x - a|$ is part of the $\delta$ statement, the question is how big can $|x + a|$ be?

Let's choose $\delta \leq 1$.

That means: $|x - a| < 1$ or equivalently:

$$-1 < x - a < 1$$

$$a - 1 < x < a + 1\,;$$ now add "a" to the entire inequality:

$$2a - 1 < x + a < 2a + 1$$

$$-2|a| - 1 \leq 2a - 1 < x + a < 2a + 1 \leq 2|a| + 1$$ so

$$|x + a| < 2|a| + 1.$$

This now means that:

$$|x^2 - a^2| = |x + a||x - a| < (2|a| + 1)|x - a|.$$

So if we can ensure that $(2|a| + 1)|x - a| < \epsilon$ or equivalently:

$$|x - a| < \frac{\epsilon}{2|a|+1}$$

we'll be in business.

So just let $\delta = \min(1, \frac{\epsilon}{2|a|+1})$ (notice that $\delta$ depends on both "a" and $\epsilon$).

Now let's show that this $\delta$ works:

Given that $|x - a| < \delta$ we know that :

$$|x^2 - a^2| = |x + a||x - a| \leq (2|a| + 1)|x - a| \quad \text{(since } \delta \leq 1)$$

$$< (2|a| + 1)\delta$$

$$\leq (2|a| + 1)(\frac{\epsilon}{2|a|+1}) = \epsilon \quad \text{(since } \delta \leq \frac{\epsilon}{2|a|+1}).$$

Hence $\lim_{x \to a} x^2 = a^2$, and $f(x) = x^2$ is continuous at $x = a$.

Ex. Let $f(x) = x^2$ if $x \neq 0$

$= 4$ if $x = 0$.

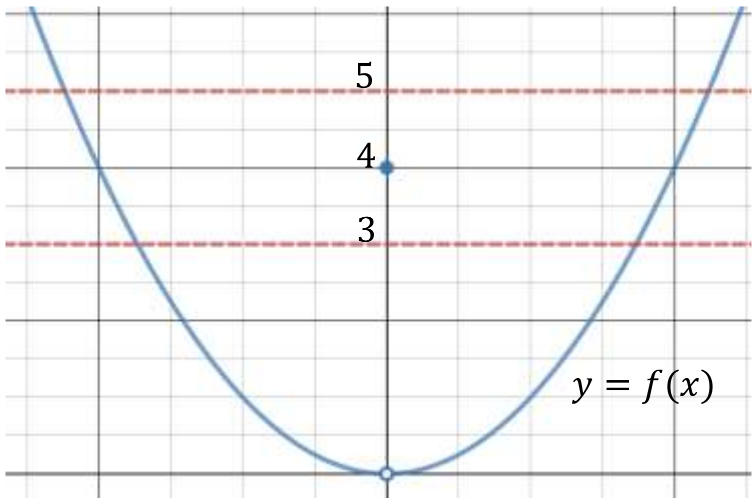
a. Using a $\delta, \epsilon$ argument prove that $f(x)$ is discontinuous at $x = 0$ (i.e. prove that $\lim_{x \to 0} f(x) \neq f(0) = 4$.)

b. Prove that $f(x)$ is not continuous on $\mathbb{R}$ by finding an open set $U$ such that $f^{-1}(U)$ is not open.

c. Prove that $f(x)$ is not continuous on $\mathbb{R}$ by finding an closed set $W$ such that $f^{-1}(W)$ is not closed.

a. We need to show that there exists an $\epsilon > 0$ such that no matter how small $\delta > 0$ is, $0 < |x - 0| < \delta$ does not imply $|x^2 - 4| < \epsilon$.

Choose $\epsilon = 1$. (We want $\epsilon$ to be less than |actual limit-value of function|)




We need to show that no matter how small $\delta > 0$ is, $0 < |x - 0| < \delta$ does not imply $|x^2 - 4| < 1$ ie, $|x^2 - 4| \geq 1$, for at least one $x$ with $0 < |x| < \delta$.

Notice that by the triangle inequality: $|-4| \leq |x^2 - 4| + |-x^2|$

Since: $|a + b| \leq |a| + |b|$; let $a = x^2 - 4$, $b = -x^2$, $a + b = -4$.

This inequality is the same as: $4 \leq |x^2 - 4| + |x^2|$

or $4 - x^2 \leq |x^2 - 4|$.

If $\delta \leq 1$ then $|x - 0| = |x| < \delta \leq 1$ and thus $|x^2| = x^2 < 1$; So we have:

$$3 < 4 - x^2 \leq |x^2 - 4|.$$

And since $\epsilon = 1$: $\epsilon = 1 < 3 < 4 - x^2 \leq |x^2 - 4|$.

So if $\delta \le 1$ every $x$ where $0 < |x| < \delta$, has $|x^2 - 4| > \epsilon = 1$.

If $\delta > 1$ then $\{x|\ |x| < 1\}$ is contained in the set of $x$, where $0 < |x| < \delta$. Thus the set points where $\delta > 1$ contains points where $|x^2 - 4| > \epsilon = 1$.

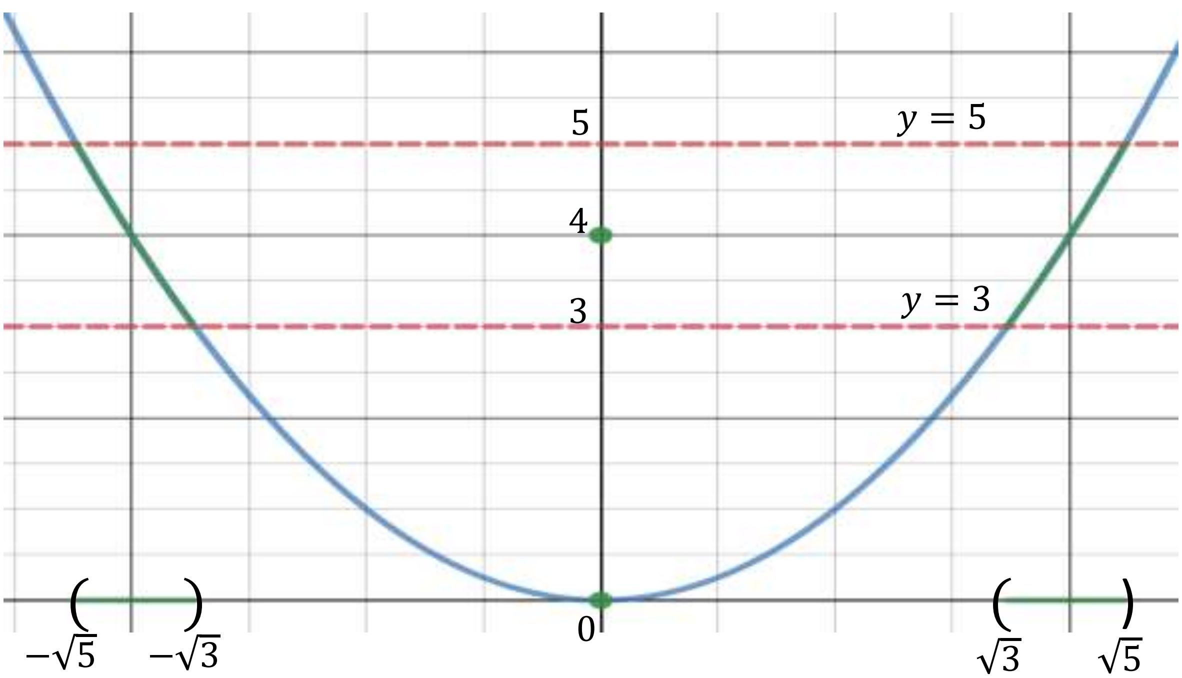
So $f(x)$ is discontinuous at $x = 0$.

b. We need to show we can find an open set $U \subseteq \mathbb{R}$ such that $f^{-1}(U)$ is not open.

We want to choose the set $U$ so that it includes the "jump" value (in this case $f(0) = 4$) but not the point $0 = \lim_{x\to 0} f(x)$. Let's take $U = (3,5)$, for example.

$f^{-1}(U) = \{x|\ 3 < f(x) < 5\}$= $\{x|\ 3 < x^2 < 5,\ x \ne 0\} \cup \{0\}$

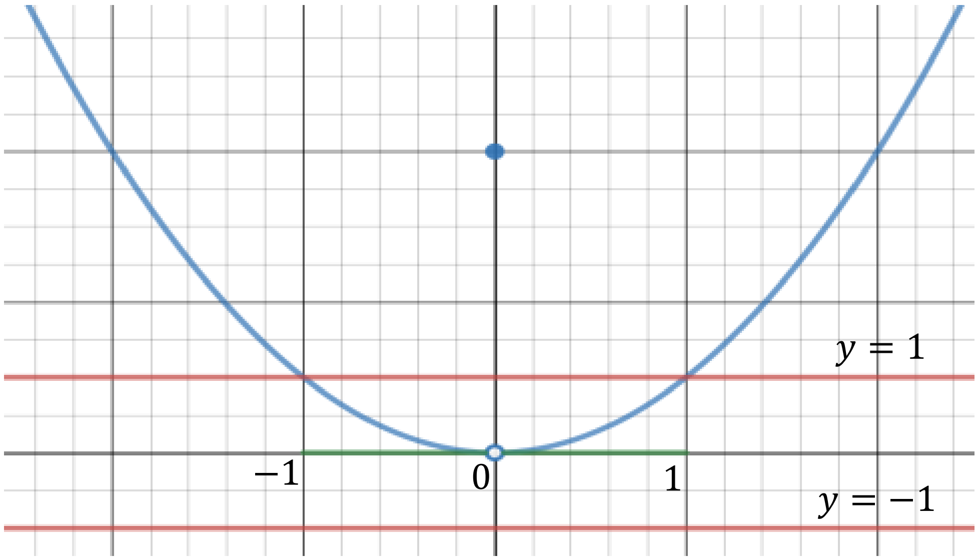
$f^{-1}(U) = \{\sqrt{3} < x < \sqrt{5}\} \cup \{-\sqrt{5} < x < -\sqrt{3}\} \cup \{0\}$




$f^{-1}(U)$ is not open because $\{0\}$ is not an interior point of $f^{-1}(U)$ (for example, there is no neighborhood of $\{0\}$ that lies totally inside of $f^{-1}(U)$ ).

c. We need to find a closed set $W \subseteq \mathbb{R}$ such that $f^{-1}(W)$ is not closed.

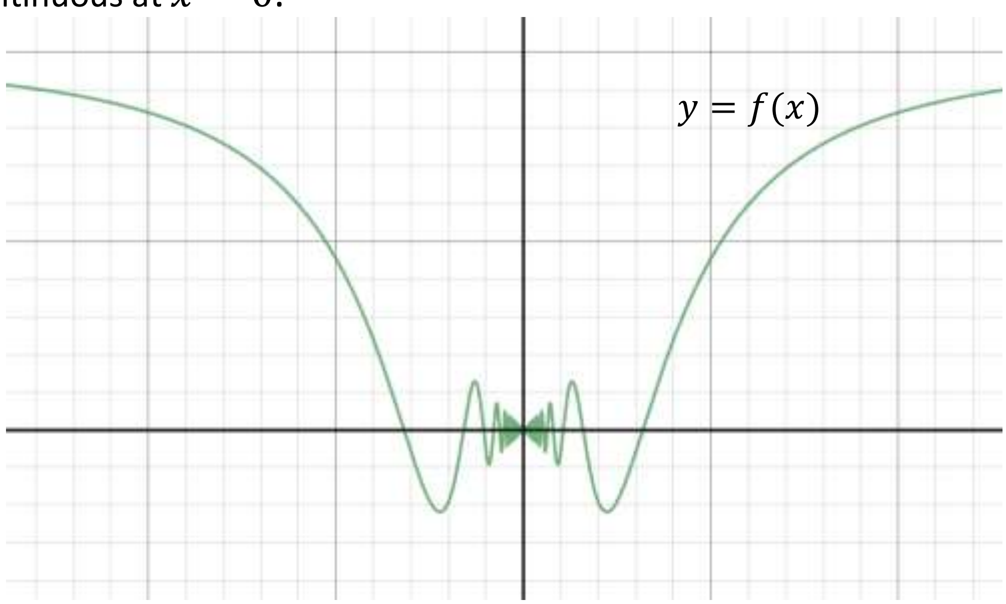
Let $W = [-1, 1]$. Then $f^{-1}(W) = [-1, 0) \cup (0, 1]$ which is not closed in $\mathbb{R}$.




Ex. Prove using a $\delta, \epsilon$ argument that $f(x) = x\sin(\frac{1}{x}) \qquad x \neq 0$

$$= 0 \qquad x = 0$$

is continuous at $x = 0$.



We must show that given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - 0| < \delta$ then $|f(x) - 0| < \epsilon$; i.e., if $|x| < \delta$ then $|f(x)| < \epsilon$.

We only need to worry about where $f(x) = xsin(\frac{1}{x})$ since at $x = 0$, $|f(0)| < \epsilon$.

Let's start with the $\epsilon$ statement:

$|xsin(\frac{1}{x})| = |x||sin(\frac{1}{x})| \leq |x| < \delta$ (since $|\sin(b)| \leq 1$ for all $b \in \mathbb{R}$)

Let $\delta = \epsilon$.

Then $|x| < \delta$ implies that:

$|xsin(\frac{1}{x}) - 0| = |xsin(\frac{1}{x})| = |x||sin(\frac{1}{x})| \leq |x| < \delta = \epsilon$.

So if $|x - 0| < \delta$ then $|f(x) - 0| < \epsilon$.

Hence $\lim_{x \to 0} f(x) = f(0)$ and $f(x)$ is continuous at $x = 0$.

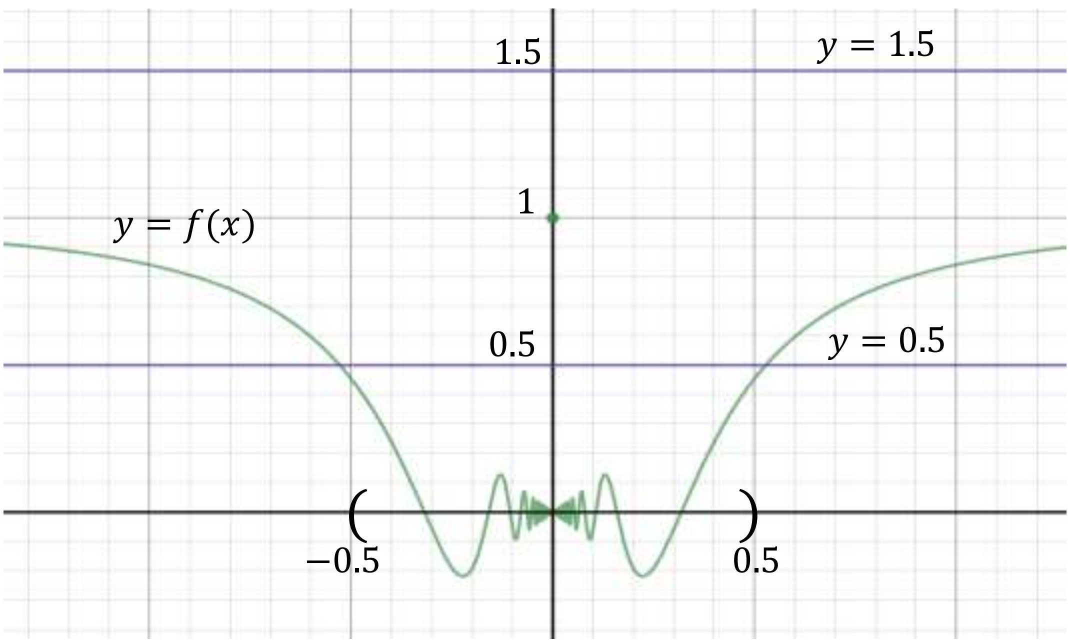
Ex. Let $f(x) = x\sin(\frac{1}{x}) \qquad x \neq 0$

$\qquad\qquad = 1 \qquad\qquad x = 0.$

Prove that $f(x)$ is discontinuous at $x = 0,$ using a $\delta, \epsilon$ argument.

We need to show that there exists an $\epsilon > 0$ such that no matter how small $\delta > 0$ is, $0 < |x - 0| < \delta$ does not imply $\left|x\sin(\frac{1}{x}) - 1\right| < \epsilon$ .

Choose $\epsilon = 1/2$ ($\epsilon = \frac{1}{2}$ is less than |actual limit-value of function|)




We need to show that no matter how small $\delta > 0$ is, $0 < |x - 0| < \delta$ does not imply $\left|x\sin(\frac{1}{x}) - 1\right| < \frac{1}{2}$ i.e., $\left|x\sin(\frac{1}{x}) - 1\right| \geq \frac{1}{2}$, for at least one $x$ with $|x| < \delta$.

In fact, we'll show that $\left|x\sin(\frac{1}{x}) - 1\right| \geq \frac{1}{2}$ for all $x$ with $0 < |x| < \delta$, for a given $\delta$.

By the triangle inequality we have:

$$|-1| \leq \left|x sin\left(\frac{1}{x}\right) - 1\right| + |-x sin\left(\frac{1}{x}\right)|$$

Since: $|a + b| \leq |a| + |b|$;

$a = x sin\left(\frac{1}{x}\right) - 1, \;\; b = -x sin\left(\frac{1}{x}\right), \;\; a + b = -1.$

$$1 \leq \left|x sin\left(\frac{1}{x}\right) - 1\right| + |x sin\left(\frac{1}{x}\right)|$$

$$1 - |x sin\left(\frac{1}{x}\right)| \;\leq \left|x sin\left(\frac{1}{x}\right) - 1\right|$$

Assume $\delta \leq \frac{1}{2}$; then $\left|x sin\left(\frac{1}{x}\right)\right| \leq |\mathrm{x}| < \frac{1}{2}$.

This means that for $|\mathrm{x}| < \frac{1}{2}$:

$$\epsilon = \frac{1}{2} < 1 - |x sin\left(\frac{1}{x}\right)| \;\leq \left|x sin\left(\frac{1}{x}\right) - 1\right|.$$

If $\delta > \frac{1}{2}$, then $\{x|\; |x| < \frac{1}{2}\}$ is contained in the set of $x$, where $|x| < \delta$. Thus the set points where $\delta > \frac{1}{2}$ contains points where $\left|x sin(\frac{1}{x}) - 1\right| > \epsilon = \frac{1}{2}$.

So $f(x)$ is discontinuous at $x = 0$.

Theorem: Let $f$ and $g$ be continuous functions from a metric space $X$ into $\mathbb{R}$ (or the complex numbers). Then $f+g,\ fg,\ \frac{f}{g}\ and\ (where\ g(x) \neq 0)$ are continuous on $X$.

Proof: At any isolated point $p \epsilon X$, we know we can find a neighborhood of $p$ that does not intersect $X$ in any other point than $p$.

Thus there exists a $\delta > 0$ such that if $d(p,x) < \delta$ then $x = p$. Hence for that $\delta$, $|h(x) - h(p)| = |h(p) - h(p)| = 0 < \epsilon$ (here $h$ represents any of

$f + g, fg, and\ \frac{f}{g}\ (where\ g(x) \neq 0$)).

At a limit point of $p \epsilon X$ since $f$ and $g$ are continuous we have:

$$\lim_{x \to p} f(x) = f(p) \quad \text{and} \quad \lim_{x \to p} g(x) = g(p).$$

By an earlier limit theorem we have:

$$\lim_{x \to p}(f(x) + g(x)) = f(p) + g(p)$$

$$\lim_{x \to p} f(x)g(x) = f(p)g(p)$$

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{f(p)}{g(p)}; \quad g(x) \neq 0; \quad g(p) \neq 0.$$

Since $f(x) = x$ and $f(x) = constant$ are continuous functions, the above theorem implies that all polynomials and rational functions where the denominator is non-zero are continuous.

Theorem: a. Let $f_1(x),\ f_2(x), f_3(x), \dots, f_k(x)$ be real valued functions on a metric space $X$, and let $f$ be a mapping of $X \to \mathbb{R}^k$ by
$f(x) = (f_1(x),\ f_2(x), f_3(x), \dots, f_k(x))$; $x\epsilon X$ then $f$ is continuous if and only if each $f_1(x), f_2(x), f_3(x), \dots, f_k(x)$ is continuous.

b. If $f, g: X \to \mathbb{R}^k$ are continuous, then $f + g$ and $f \cdot g$ are continuous.

Proof: a. Assume $f: X \to \mathbb{R}^k$ is continuous at $x = p$ and show $f_i(x), i = 1, \dots, n$ are continuous at $x = p$.

So for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $d(x,p) < \delta$ then $d\big(f(x), f(p)\big) < \epsilon$. That is:

$$\left(\sum_{i=1}^{n}\big(f_i(x) - f_i(p)\big)^2\right)^{\frac{1}{2}} < \epsilon .$$

However, notice that

$$|f_i(x) - f_i(p)| \leq \left(\sum_{i=1}^{n}\big(f_i(x) - f_i(p)\big)^2\right)^{\frac{1}{2}} < \epsilon .$$

So the same $\delta$ that forces $d\big(f(x), f(p)\big) < \epsilon$ will force $d\big(f_i(x), f_i(p)\big) < \epsilon$,

Thus $f_i(x), i = 1, \dots, n$ are continuous at $x = p$.

Now assume $f_i(x), i = 1, \dots, n$ are continuous and show $f(x)$ is continuous.

So for all $\epsilon > 0$ there exists a $\delta_i > 0$ such that if $d(x,p) < \delta_i$ then $d\big(f_i(x), f_i(p)\big) < \epsilon/n$.

Choose $\delta = \min(\delta_1, \dots, \delta_n)$ and notice that:

$$\left(\sum_{i=1}^{n}\big(f_i(x) - f_i(p)\big)^2\right)^{\frac{1}{2}} \leq \sum_{i=1}^{n}|f_i(x) - f_i(p)| < n\left(\frac{\epsilon}{n}\right) = \epsilon.$$

Thus $f(x)$ is continuous at $x = p$.

b. Follows from part a and the continuity theorem on page 16.