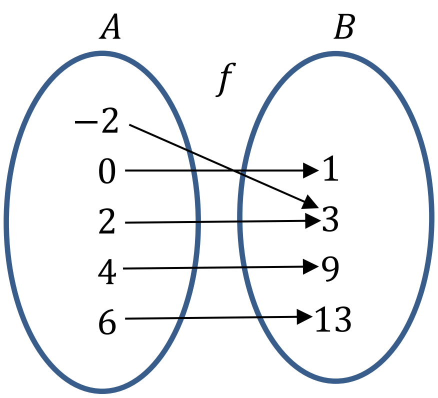
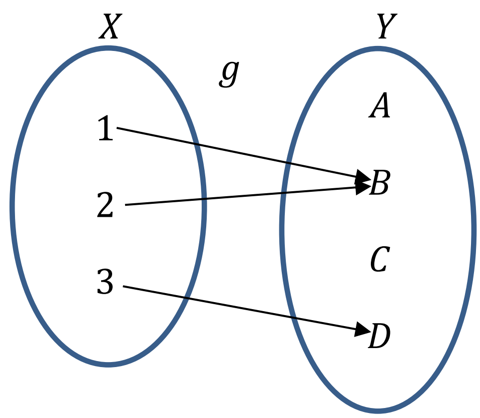
Finite, Countable, and Uncountable Sets

Def. Let A and B be 2 sets and f a mapping of A into B. If $E \subseteq A$, $f(E)$ is defined to be the set of elements $f(x)$, for $x\epsilon E$. $f(E)$ is called the **image** of E under f. If $f(A) = B$, we say that f **maps A onto B**.

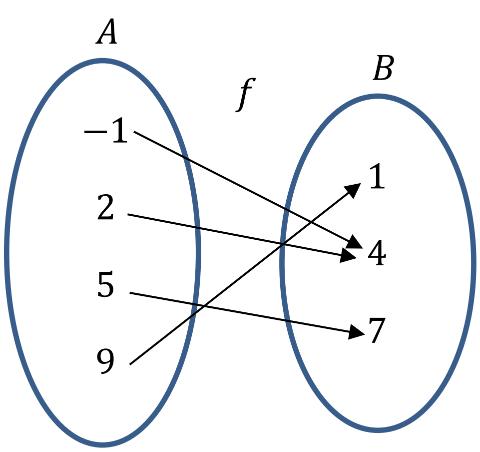
f maps A onto B



g maps X into Y



Def. If $C \subseteq B$, $f^{-1}(C)$ is the set of all $x\epsilon A$ such that $f(x)\epsilon C$. We call $f^{-1}(C)$ the **inverse image** of C under f. If $y\epsilon B$, $f^{-1}(y)$ is the set of all $x\epsilon A$ such that $f(x) = y$.

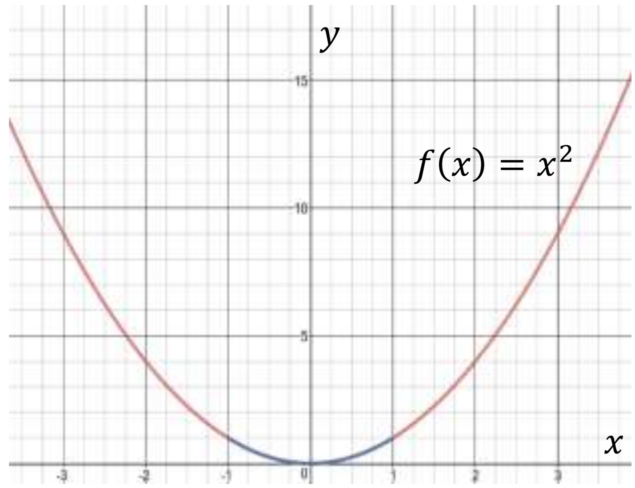


$f^{-1}(4) = \{-1,2\}$

$f^{-1}(\{1,4\}) = \{-1,2,9\}$

Ex. Let $A = \mathbb{R}$ and $B = \mathbb{R}$, and $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Now let

$E = [-1,1] \subseteq A$, and $C = [1,9] \subseteq B$. Find $f(E)$, $f(A)$, $f^{-1}(C)$, and $f^{-1}(4)$.



$f(E) = [0,1]$ and $f(A) = [0, \infty)$.

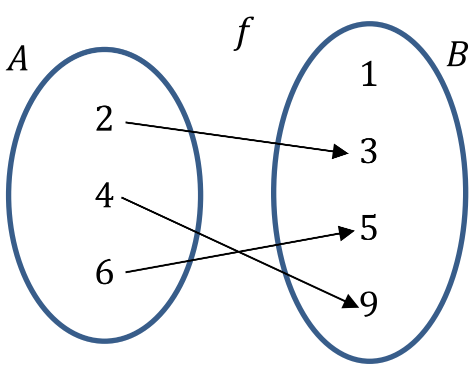
$f^{-1}(C) = \{x | f(x) \epsilon [1,9]\} = \{x | 1 \le x^2 \le 9\}$

$= \{-3 \le x \le -1\} \cup \{1 \le x \le 3\}$

$f^{-1}(4) = \{x | x^2 = 4\} = \{-2, 2\}$

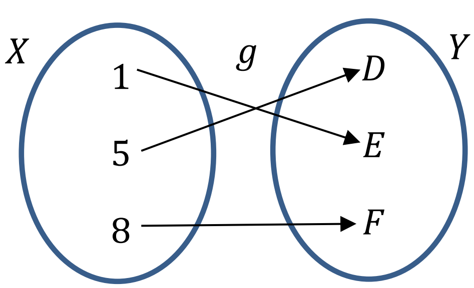
Def. Let $f: A \to B$. If for each $y \epsilon B$, $f^{-1}(y)$ consists of at most 1 element, then f is said to be a **1-1 mapping** of A into B.

Note: this is the same as saying that f is 1-1 if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.



f is 1-1 from A into B.

Def. If there exists a 1-1 mapping of A onto B, then we say that A and B can be put into **1-1 correspondence** or that $\boldsymbol{A \sim B}$, i.e., **A is equivalent to B**.



g is 1-1 from X onto Y, so $X \sim Y$.

Ex. Let $A = \mathbb{R}$ and $B = \mathbb{R}$, and $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. $f(x)$ is NOT a 1-1 mapping of A into B since, as we just saw, $f^{-1}(4) = \{-2, 2\}$. To be 1-1, the inverse image of every point in B can have at most 1 point (it can have 0 points). However, if $A = \mathbb{R}^+ \cup \{0\} = \{x \in \mathbb{R} |\ x \geq 0\}$, then $f: A \to \mathbb{R}$ is 1-1 into $\mathbb{R}$.

Ex. Let $A = \{2,4,6,8\}$, $B = \{1,3,5,7,9\}$, $C = \{1,3,5,7\}$. Let's define the following mappings:

$f: A \to B$ by $f(2) = 1$, $f(4) = 3$, $f(6) = 5$, $f(8) = 7$

$g: A \to C$ by $g(2) = 1$, $g(4) = 3$, $g(6) = 5$, $g(8) = 7$

Notice that both f and g are 1-1, but f is 1-1 from A into B and g is 1-1 from A onto C. So we can say that $A \sim C$, A is equivalent to C, but A is not equivalent to B (A is not equivalent to B because we can't find a 1-1 mapping from A onto B).

Def. Let $J_n = \{1,2,3,4, \dots n\}$; and $J = \{1,2,3,4 \dots\}$ (i.e., J is all positive integers).

a. A is called **Finite** if $A \sim J_n$ for some n (the empty set is also considered finite).

b. A is called **Infinite** if A is not finite

c. A is called **Countable** (or countably infinite) if $A \sim J$

d. A is called **Uncountable** (or uncountably infinite) if A is neither Finite nor Countable

e. A is called **at most countable** if A is Finite or Countable

Note: Countable sets are also sometimes called enumerable or denumerable.

These definitions give us a way to talk about the "size" of an infinite set. We say that 2 infinite sets are equivalent (or have the same "size") if we can find a 1-1 correspondence from one set onto the other set. This leads to some counterintuitive results. For example, we can have one infinite set be a proper subset of the other, yet they have the same “size” (clearly, this can’t happen for a finite set).

Ex. Let $A = \{1,2,3,4,\ldots\}$, and $B = \{0,1,2,3,4,\ldots\}$. Even though $A \subseteq B$ and A is a proper subset of B we can find a 1-1 correspondence of A onto B:

$$f: A \to B; \text{ by } \quad f(x) = x - 1.$$

We need to show that f is 1-1 and onto.

If $f(p) = f(q)$ then $p - 1 = q - 1$, thus $p = q$ and f is 1-1.

Choose any $y \epsilon B$. To show f is “onto” we must be able to find an $x \epsilon A$ such that $f(x) = y$.

But $f(x) = x - 1 = y$ implies that $x = y + 1$ and $(y + 1) \epsilon A$, so $f(x)$ is also “onto”.

Thus A is equivalent to B (ie they have the same “size”). They are both countably infinite.

Ex. Let $A = \{1,2,3,4,\ldots\}$, and $B = \{2,4,6,8,\ldots\}$. Show that $A \sim B$.

Let $f: A \to B$; by $f(x) = 2x$.

We need to show that f is 1-1 and onto.

Notice that if $f(p) = f(q)$ then $2p = 2q$ and thus $p = q$. So f is 1-1.

To show f is “onto” we must show given a $y \epsilon B$, we can find an $x \epsilon A$ such that $f(x) = y$.

$f(x) = 2x = y$ implies that $x = \frac{y}{2}$, which is in A. Thus f is “onto” and $A \sim B$.

Ex. $A = \{1,2,3,4, \ldots\}$, and $B = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ (i.e., B is all of the integers). Show A and B are again equivalent:

Let $f: A \to B$; by $f(x) = \frac{x}{2}$ if x is even

$= -(\frac{x-1}{2})$ if x is odd

We need to show that f is 1-1 and onto.

If $f(p) = f(q)$ and p is even, then $f(p) = \frac{p}{2} > 0$.

Hence $f(q) = \frac{q}{2}$, otherwise $f(q) \leq 0$ and can't equal $f(p) = \frac{p}{2} > 0$.

Thus $\frac{p}{2} = \frac{q}{2}$, and $p = q$.

Similarly, if p is odd $f(p) = -\left(\frac{p-1}{2}\right) \leq 0$, and $f(q) = -\left(\frac{q-1}{2}\right) \leq 0$.

Once again, $-\left(\frac{p-1}{2}\right) = -\left(\frac{q-1}{2}\right)$, and $p = q$, so f is 1-1.

To show f is "onto" we need to show given any $y \epsilon B$, we can find an $x \epsilon A$ such that $f(x) = y$.

If $y \leq 0$ then $f(x) = -\left(\frac{x-1}{2}\right) = y$.

Solving for x we get $x = -2y + 1$ which is in A, and $f(-2y + 1) = y$.

If $y > 0$ then $f(x) = \left(\frac{x}{2}\right) = y$.

Solving for x we get $x = 2y$, which is in A, and $f(2y) = y$.
So f is onto and $A \sim B$.

Theorem: The set of positive rational numbers, $\mathbb{Q}^+$, is countable.

Proof: One can put the set of positive rational numbers, $\mathbb{Q}^+$, into 1-1 correspndence with the positive integrers, $\mathbb{Z}^+$, by creating a "large" table with the positive integers along the top and side and taking their ratios as the entries of the table. One then creates a 1-1 mapping with $\mathbb{Z}^+$ by matching the positive integers with elements of the table by taking longer and longer diagonals and "throwing out" duplicate rational numbers (the red numbers):

	1	2	3	4	5	6	7
1	1/1	1/2	1/3	1/4	1/5	1/6	1/7
2	2/1	2/2	2/3	2/4	2/5	2/6	2/7
3	3/1	3/2	3/3	3/4	3/5	3/6	3/7
4	4/1	4/2	4/3	4/4	4/5	4/6	4/7
5	5/1	5/2	5/3	5/4	5/5	5/6	5/7
6	6/1	6/2	6/3	6/4	6/5	6/6	6/7
7	7/1	7/2	7/3	7/4	7/5	7/6	7/7
8	8/1	8/2	8/3	8/4	8/5	8/6	8/7

$1 \rightarrow \frac{1}{1}$, $2 \rightarrow \frac{2}{1}$, $3 \rightarrow \frac{1}{2}$, $4 \rightarrow \frac{3}{1}$, $5 \rightarrow \frac{1}{3}$, etc.

Actually, the set of all rational numbers is countable.

A similar argument shows a countable union of countable sets is countable.

Theorem: The set of real numbers between 0 and 1 (inclusive) is uncountable.

Proof: Let's assume that we can list (listing is the same as creating a 1-1 map with the positive integers) all of these real numbers and get a contradiction.

$x_1 = 0.\,a_{11}a_{12}a_{13}a_{14}\ldots$

$x_2 = 0.\,a_{21}a_{22}a_{23}a_{24}\ldots$

$x_3 = 0.\,a_{31}a_{32}a_{33}a_{34}\ldots$

$x_4 = 0.\,a_{41}a_{42}a_{43}a_{44}\ldots$

$\vdots$

where a_{ij} is an integer with $0 \leq a_{ij} \leq 9$.

But we can always create a real number, x, between 0 and 1 inclusive, which is not on this list by:

$x = 0.\,{\sim}a_{11}{\sim}a_{22}{\sim}a_{33}{\sim}a_{44}\ldots$.

Where ${\sim}a_{ii}$ means any digit other than a_{ii}.

That contradicts the assumption that we could list all real numbers between 0 and 1. Hence this set is uncountable.

Ex. Show the set of real numbers between 0 and 5 is equivalent to the set of real numbers between 0 and 1.

Let $A = \{x \epsilon \mathbb{R} |\ 0 \leq x \leq 1\}$ and $B = \{x \epsilon \mathbb{R} |\ 0 \leq x \leq 5\}$.

Define $f: A \to B$; by $f(x) = 5x$.

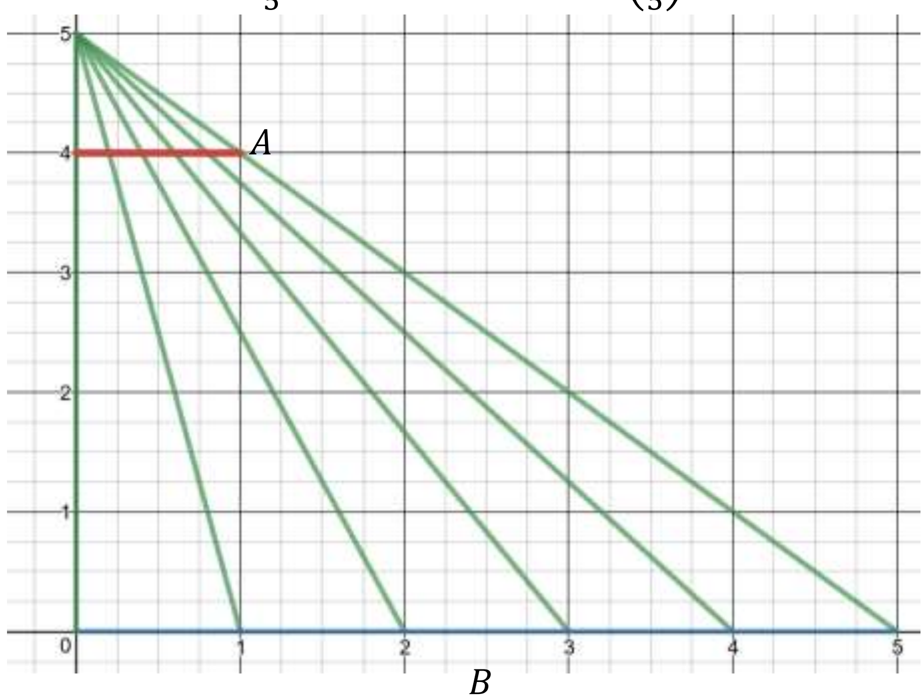
We must show that f is a 1-1 mapping of A onto B.

To show f is 1-1: $f(p) = f(q)$ implies that $5p = 5q$ and $p = q$.

To show f is onto: given a $y \epsilon B$, we can find an $x \epsilon A$ such that $f(x) = y$.

$f(x) = 5x = y$, solving for x we get: $x = \frac{y}{5}$, which is in A, and $f\left(\frac{y}{5}\right) = y$.

So f is onto and $A \sim B$.



Ex. Show that the set of real numbers strictly between 0 and 1 is equivalent to the set of all positive real numbers.

Let $A = \{x\epsilon\mathbb{R} |\ 0 < x < 1\}$ and $B = \{x\epsilon\mathbb{R} |\ 0 < x\}$.

Define $f: A \to B$; by $f(x) = \frac{x}{1-x}$

$f(x)$ is 1-1 since if $f(p) = f(q)$ then $\frac{p}{1-p} = \frac{q}{1-q}$.

$p(1-q) = q(1-p)$

which implies $p = q$.

$f(x)$ is onto since given any positive real number y,

$f(x) = \frac{x}{1-x} = y$

$x = y(1-x) = y - xy$

$x + xy = y$

$x(1+y) = y$

$x = \frac{y}{1+y} \in A$, and $f\left(\frac{y}{1+y}\right) = y$

Thus $A \sim B$.

Def. $S = \bigcup_{m=1}^{n} E_m = E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n$

$T = \bigcup_{m=1}^{\infty} E_m = E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n \cup \ldots$

means x is a member of S (or T) if and only if $x\epsilon E_i$ for some i.

Ex. Let $E_i = [i, i+1]$; where i is a positive integer

i.e. $E_i = \{x \epsilon \mathbb{R} |\ i \leq x \leq i+1,\ where\ i\ is\ a\ positive\ integer\}$

For example, $E_5 = \{x \epsilon \mathbb{R} |\ 5 \leq x \leq 6\}$. Find $\bigcup_{m=1}^{n} E_m$, $\bigcup_{m=1}^{\infty} E_m$.

$\bigcup_{m=1}^{n} E_m = E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n = [1, n+1]$

$\bigcup_{m=1}^{\infty} E_m = E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n \cup \ldots = [1, \infty)$

Def. $P = \bigcap_{i=1}^{n} E_i = E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n$

$Q = \bigcap_{i=1}^{\infty} E_i = E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n \cap \ldots$

means x is a member of P (or Q) if and only if $x \epsilon E_i$ for all $i = 1,2,3, \ldots n$ (or ∞)

Ex. Let $E_i = [i, \infty)$, where $i \in \mathbb{Z}^+$. Find $\bigcap_{i=1}^{n} E_i$, $\bigcap_{i=1}^{\infty} E_i$.

$\bigcap_{i=1}^{n} E_i = E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n = [1, \infty) \cap [2, \infty) \cap \cdots \cap [n, \infty) = [n, \infty)$

$\bigcap_{i=1}^{\infty} E_i = E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n \cap \ldots = \emptyset$ (the empty set).

Ex. Let $F_i = [0, \frac{1}{i}]$; i a positive integer, find $\bigcap_{i=1}^{10} F_i$, $\bigcap_{i=1}^{\infty} F_i$.

$$\bigcap_{i=1}^{10} F_i = F_1 \cap F_2 \cap F_3 \cap \ldots \cap F_{10} = [0,1] \cap \left[0, \frac{1}{2}\right] \cap \left[0, \frac{1}{3}\right] \cap \ldots \cap \left[0, \frac{1}{10}\right]$$

$$= \left[0, \frac{1}{10}\right].$$

$\bigcap_{i=1}^{\infty} F_i = F_1 \cap F_2 \cap F_3 \cap \ldots \cap F_{10} \cap \ldots = \{0\}$.