Homomorphisms and Factor Rings- HW Problems

1. Let n be a positive integer. For what values of n does $\mathbb{Z}_n$ contain a subring isomorphic to $\mathbb{Z}_2$?

2. Write down the addition and multiplication tables for $2\mathbb{Z}/12\mathbb{Z}$.

   Are $2\mathbb{Z}/12\mathbb{Z}$ and $\mathbb{Z}_6$ isomorphic rings?

In problems 3-8 determine whether the statement is true or false.

3. If $\phi: R \to R'$ is a ring homomorphism then $\phi: (R, +) \to (R', +')$ is a group homomorphism.

4. If $\phi: (R, +) \to (R', +')$ is a group homomorphism, where R and R' are rings then $\phi: R \to R'$ is a ring homomorphism.

5. If $\phi: R \to R'$ is a ring homomorphism and $\ker(\phi) \neq \{0\}$ then ϕ is not one-to-one.

6. $\mathbb{Z}$ is an ideal in $\mathbb{Q}$.

7. If R' is a subring of R then R' is an ideal of R.

8. The rings $\mathbb{Z}/8\mathbb{Z}$ and $\mathbb{Z}_8$ are isomorphic rings.

9. Let $R = \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$ and R' the 2×2 matrices $\begin{bmatrix} a & 3b \\ b & a \end{bmatrix}$ where $a, b \in \mathbb{Z}$.

a. Show that R is a subring of $\mathbb{R}$.

b. Show that R' is a subring of $M_2(\mathbb{Z})$.

c. Show that $\phi: R \to R'$ by $\phi(a + b\sqrt{3}) = \begin{bmatrix} a & 3b \\ b & a \end{bmatrix}$ is a ring isomorphism.

10. Let ϕ be a homomorphism from a field F to a ring R. Show that either ϕ is one-to-one or $\phi[F] = \{0\}$.

11. Suppose $\phi_1: R_1 \to R_2$ and $\phi_2: R_2 \to R_3$ are ring homomorphisms. Prove that $\phi_2 \circ \phi_1: R_1 \to R_3$ is a ring homomorphism.

12. Suppose $\phi: R \to R'$ is a ring homomorphism of a ring with unity, R, onto a non-zero ring R'. Let u be a unit in R. Prove that $\phi(u)$ is a unit in R'.